On a Nonlinear Binomial Equation of Third Order

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A necessary and sufficient condition for the solution of equation $u^{\prime\prime\prime} + p(t)u^{\alpha} = 0$ ($\alpha > 0$ an odd integer, $p \leq 0$ on $(a, \infty)$) to be oscillatory and some sufficient conditions for the solution in the cases $p \leq 0$ and $p \geq 0$ to be oscillatory or non-oscillatory are derived. For this methods and results of the theory of linear differential equations of the third order are effectively used.

Key words: Third order nonlinear differential equations, oscillatory solutions, non-oscillatory solutions, bounded solutions

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1. The paper investigates properties of solutions of the binomial differential equation of third order $u^{\prime\prime\prime} + pu^{\alpha} = 0$, (1)

where $p$ is a continuous function on the interval $(a, \infty)$ with $a > -\infty$, and $\alpha > 1$ is an odd number. Some of our results can be generalized to the case where $\alpha$ is a ratio of odd integers. The problem has already been a research object of many authors, see [1, 3-6] and others. Here the methods developed in the study of linear differential equation of third order [2] are effectively used.

2. By a solution of equation (1) we mean a function $u$ defined on a subinterval $\mathcal{I} \subset (a, \infty)$, with continuous third derivative and satisfying equation (1). By an oscillatory solution of equation (1) we mean a solution $u$ of (1) that has on the interval $\mathcal{I}$ infinitely many null points, with a limit point at the right end point of the interval $\mathcal{I}$. Otherwise the solution is called non-oscillatory. A non-extentable solution $u$ defined on a bounded from above interval $\mathcal{I}$ is sometimes called singular.

Equation (1) can be written in the linear form

$u^{\prime\prime\prime} + pu^{\alpha-1}u = 0$. (1)*

The adjoint equation to (1)* has the form

$v^{\prime\prime\prime} - pu^{\alpha-1}v = 0$. (2)

Let $t_0 \in \mathcal{I}$ and let $u$ be a solution of equation (1) with the property $u(t_0) = u_0, u'(t_0) = u'_0, u''(t_0) = u''_0$, where at least one of the numbers $u_0, u'_0, u''_0$ is non-zero. Further, let $v$ be a solution of equation (2) with the property $v(t_0) = v_0, v'(t_0) = v'_0, v''(t_0) = v''_0$, where again at least one of the numbers $v_0, v'_0, v''_0$ is non-zero. Then for $t \in \mathcal{I}$ we have (see [2])

$v(t)u''(t) - v'(t)u'(t) + v''(t)u(t) = \text{const}$, (3)

where $\text{const} = v_0u''_0 - v'_0u'_0 + v''_0u_0$. 
If we multiply equation (1) by the solution \( u \) and integrate from \( t_0 \) to \( t \in \mathcal{I} \), then we obtain for all \( t \in \mathcal{I} \) the integral identity

\[
\int_{t_0}^{t} \left(u(t)u'(t) - \frac{1}{2} u^2(t) - \int_{t_0}^{t} \rho(t)u^{\alpha-1}(t)u^\gamma(t)\,dt\right)\,dt = \text{const.} \tag{4}
\]

Similarly, for equation (2) we obtain for all \( t \in \mathcal{I} \)

\[
\int_{t_0}^{t} \left(v(t)v'(t) - \frac{1}{2} v^2(t) - \int_{t_0}^{t} \rho(t)u^{\alpha-1}(t)v^\gamma(t)\,dt\right)\,dt = \text{const.} \tag{5}
\]

**Corollary 1:** Let \( p \geq 0 \) \( (p \leq 0) \) on \((a, \infty)\) and \( p \neq 0 \) on any subinterval of \((a, \infty)\). Further, let \( u \) be a solution of equation (1) defined on an interval \( \mathcal{I} \subset (a, \infty) \) and with the property \( u(t_0) = u'(t_0) = 0 \) for some \( t_0 \in \mathcal{I} \). Then \( u(t) \neq 0 \), \( u'(t) \neq 0 \) for all \( t < t_0 \) \((t > t_0)\).

A similar assertion holds for the solution \( v \) of the equation (2) with the property \( v(t_0) = v'(t_0) = 0 \), \( v''(t_0) \neq 0 \) for some \( t_0 \in \mathcal{I} \), that is \( v(t) \neq 0 \), \( v'(t) \neq 0 \), \( v''(t) \neq 0 \) for all \( t > t_0 \) \((t < t_0)\).

**Proof:** It follows from the identities (4) and (5) and from the equations (1) and (2), respectively.

**Corollary 2:** Supposing \( p \) is the same as in Corollary 1, each solution \( u \) of equation (1) or (2) has at most one double null point.

3. Our goal is to derive some properties of solutions of equation (1) in the case \( p \leq 0 \).

**Theorem 1:** Let \( p \leq 0 \) on \((a, \infty)\). Then any non-extendable solution \( u \) of equation (1) defined on an interval \( \mathcal{I} \subset (a, \infty) \) and such that \( u(t_0) > 0 \), \( u'(t_0) > 0 \), \( u''(t_0) > 0 \) for some \( t_0 \in \mathcal{I} \) has the property \( u(t) > 0 \), \( u'(t) > 0 \), \( u''(t) > 0 \) for all \( t > t_0 \) and, moreover, \( u(t) \to \infty \), \( u'(t) \to \infty \) as \( t \) converges to the right end point of the interval \( \mathcal{I} \).

**Proof:** First of all we show that \( u''(t) > 0 \) for all \( t > t_0 \). Let us form the function \( V = u u'' \). If \( u'' \) has null points to the right of \( t_0 \), let us denote by \( t_1 \) the smallest of them. Hence \( u''(t_1) = 0 \). Therefore \( u(t) > 0 \), \( u'(t) > 0 \) for all \( t \in (t_0, t_1) \) and \( V(t_1) = 0 \). Since \( p \leq 0 \) there holds

\[
dV(t)/dt = u''(t)u(t) + u'(t)u''(t) - \rho(t)u^{\alpha+1}(t)u'(t) > 0 \text{ for all } t \in (t_0, t_1).
\]

After integration from \( t_0 \) to \( t_1 \) we obtain \( 0 = V(t_0) + \int_{t_0}^{t_1} V'(t)\,dt > 0 \), which is a contradiction. Hence \( u''(t) > 0 \) for all \( t > t_0 \). From here it follows that \( u(t) > 0 \), \( u'(t) > 0 \) for all \( t > t_0 \). From equation (1) it also follows that \( u''(t) \geq 0 \) for all \( t > t_0 \). From these inequalities we then have that \( u(t) \to \infty \), \( u'(t) \to \infty \) as \( t \) converges to the right end point of the interval \( \mathcal{I} \).

N. Parhi and S. Parhi have proved the following

**Theorem A [6 : Theorem 3.1]:** Let \( p \leq 0 \) and \( \int_{t_0}^{\infty} p(t)\,dt = -\infty \). Then every bounded solution of equation (1) in \((t_0, \infty)\) is oscillatory in \((t_0, \infty)\).

**Lemma 1:** Let the assumptions of Theorem A be fulfilled and let \( u \) be a solution of equation (1) with the property \( u(t) > 0 \) for all \( t \geq t_0 \), where \( t_0 > a \). Then there exists such \( t_1 > t_0 \) that \( u(t) > 0 \), \( u'(t) > 0 \), \( u''(t) > 0 \) for all \( t > t_1 \).
Proof: From equation (1) it follows that $u''(t) \geq 0$ for all $t > t_0$. Then we have two possibilities for $u''$:

1. $u''(t_0) > 0$ and hence $u''(t) > 0$ for all $t > t_0$. Then after integration of equation (1) we get

$$u''(t) = u''(t_0) - \int_{t_0}^{t} p(t) u'(t) \, dt,$$

$$u'(t) = u'(t_0) + u''(t_0) (t - t_0) - \int_{t_0}^{t} (t - \tau) p(t) u'(t) \, d\tau,$$

$$u(t) = u(t_0) + u'(t_0) (t - t_0) + u''(t_0) \frac{(t - t_0)^2}{2!} - \int_{t_0}^{t} \frac{(t - \tau)^2}{2!} p(t) u'(t) \, d\tau.$$

From the second equation of (6) the existence of such $t_1 > t_0$ follows that $u(t) > 0$ for all $t > t_1$.

2. $u''(t) < 0$ for all $t > t_0$. Then $u'$ is decreasing and there are again two possibilities:

(i) $u'(t) < 0$ for all $t > t_1$ and $u'$ decreasing. Hence $u'(t) < u'(t_1)$ from where $u(t) < u(t_1) + u'(t_1) (t - t_1)$ and this is a contradiction to the assumption that $u(t) > 0$ for all $t > t_0$.

(ii) $u'(t) > 0$ for all $t > t_0$. Then the function $u$ is decreasing in $(t_1, \infty)$ and from the assumptions on $p$ there follows that, for certain $t_1 > t_0$, $u''(t) > 0$ for all $t > t_1$ and this again leads to a contradiction to the assumption that $u'(t) < 0$ for all $t > t_0$.

The following theorem answers to the question which solutions of equation (1), under the assumptions of Theorem A, can be oscillatory.

**Theorem 2:** Let the assumptions of Theorem A concerning $p$ be fulfilled. Then a necessary and sufficient condition for a solution $u$ of equation (1) to be oscillatory for $t > t_0$, for some $t_0 > a$, is that

$$u(t) u''(t) - u''^2(t)/2 < 0 \text{ for all } t > t_0. \quad (7)$$

**Proof:** Sufficient. Let (7) hold and let e.g. $u(t) > 0$ for all $t > t_0$. It follows from Lemma 1 that there exists such $t_1 > t_0$ that $u(t_1) > 0$, $u'(t_1) > 0$, $u''(t_1) > 0$ and, from Theorem 1, $u(t) \to \infty$ as $t \to \infty$. From the integral identity (4) it follows that

$$u(t) u''(t) - u''^2(t)/2 = u(t_1) u''(t_1) - u''^2(t_1)/2 - \int_{t_1}^{t} p(t) u''(t) \, d\tau.$$

and from this and the assumptions of Theorem 2 there follows a contradiction with (7) as $t \to \infty$.

**Necessity.** Let the solution $u$ of equation (1) be oscillatory in $(t_0, \infty)$ and let $t_j$ ($j = 1, 2, \ldots$) be null points of $u$ in $(t_0, \infty)$. Then from the relation (8) it follows that the function $uu'' - u''^2/2$ is increasing in $(t_j, \infty)$, but $u(t_j) u''(t_j) - u''^2(t_j)/2 < 0$. From this fact it follows that (7) holds for all $t > t_j$.

**Theorem 3:** Suppose that $p \leq 0$ on $(a, \infty)$ and $p \equiv 0$ on any subinterval of $(a, \infty)$. Let $u$ be a solution of equation (1) defined on an interval $\mathcal{I} \subset (a, \infty)$ and satisfying $k := u(t_0) u''(t_0) - u''^2(t_0)/2 \geq 0$ for some $t_0 \in \mathcal{I}$. Then $u$ does not have a null point to the right of $t_0$ and $|u(t)| \to \infty$, $|u'(t)| \to \infty$ as $t$ converges to the right end point of $\mathcal{I}$.
Proof: The solution \( u \) fulfills the identity \((4)\), i.e.

\[
    u(t)u''(t) - u'^2(t)/2 + \int_0^t p(t)u^{\alpha+1}(t)\,dt = k > 0 \quad \text{for all } t \in \mathcal{I}.
\]

Let \( u(t_1) = 0 \) for some \( t_1 > t_0 \). Then from the identity above at the point \( t_1 \) we get a contradiction. To prove the second part of the assertion let us suppose for simplicity that \( u(t) > 0 \) for all \( t > t_0 \). Then also \( u''(t) > 0 \) for all \( t > t_0 \) and from the identity \((9)\) it follows that \( u''(t) > 0 \) for all \( t > t_0 \). Suppose that \( \mathcal{I} \) is a bounded interval with right end point \( b \) and let \( u \) be bounded on it. Then also \( u'' \) is bounded as follows from the first relation in \((6)\). Note that \( u'' \) is a monotone function. From the second relation in \((6)\) it follows that the function \( u'' \) is also monotone and bounded. Hence \( u(t) \to u_0, u'(t) \to u_0', u''(t) \to u_0'' \) as \( t \to b \), where \( u_0, u_0', u_0'' \) are real numbers. That means \( u \) can be extended to \( b \), which is a contradiction and therefore \( u(t) \to \infty, u'(t) \to \infty \) as \( t \to b \). In the case \( b = \infty \) the proof is trivial - it follows from the monotonicity of the functions \( u'', u''' \) and from \((6)\).

Theorem 4: Let \( p(t) < -k^2 \) \((k > 0)\) for all \( t > t_0 \). Then each oscillatory solution \( u \) of the equation \((1)\) defined on \((t_0, \infty)\) belongs to the class \( \mathcal{B}\alpha+1 \) on \([t_0, \infty)\), i.e. \( \int_0^\infty u^{\alpha+1}(t)\,dt < \infty \).

Proof: It follows again from the identity \((4)\). Really, from Theorem 2 it follows that

\[
    u(t)u''(t) - u'^2(t)/2 = u(t_0)u''(t_0) - u'^2(t_0)/2 - \int_{t_0}^t p(t)u^{\alpha+1}(t)\,dt < 0.
\]

This implies \( \int_{t_0}^\infty p(t)u^{\alpha+1}(t)\,dt < \infty \).

4. Now our goal is to derive properties of solutions of equation \((1)\) in the case \( p \geq 0 \). For this let \( u \) be a solution of the differential equation \((1)\) defined on an interval \( \mathcal{I} \subset (a, \infty) \) and suppose that it fulfills the initial conditions \( u(t_0) = u_0, u'(t_0) = u_0', u''(t_0) = u_0'' \) for some \( t_0 \in \mathcal{I} \). Notice that the relations \((6)\) hold.

Lemma 2: Let \( p \geq 0 \) on \((a, \infty)\) and let \( u \) be a non-extentable solution of equation \((1)\) defined on \([t_0, b)\), for some \( b \in (t_0, \infty) \). Then \( b = \infty \).

Proof: It follows from the relations \((6)\). Indeed, suppose \( b < \infty, u(t) > 0 \) for all \( t \in [t_0, b) \) and bounded from above. Then from the relations \((6)\) it follows that \( u \) can be extended to \( b \). If \( u \) is unbounded on \([t_0, b)\) and \( \int_0^b (b - t)^2 p(t)u^2(t)\,dt \) exists, then \( u \) and also \( u', u'' \) can be extended to \( b \). If \( \int_0^b (b - t)^2 p(t)u^2(t)\,dt = \infty \), then from the third relation in \((6)\) it follows that \( u \) must have a zero and this is a contradiction to \( u(t) > 0 \) for all \( t \in [t_0, b) \).

Remark 1: Lemma 2 does not hold in the case of extendability of the solution to the left of the point \( t_0 \). For example the equation \( u'' - \alpha(\alpha + 1)(\alpha + 2)\alpha^2 u' + 2\alpha^2 u = 0 \) has a solution \( u = t^{-\alpha} \) defined on \((0, \infty)\). It cannot be extended to the left of 0.

Lemma 3: Let \( p \geq 0 \) on \((a, \infty)\) and let \( u \) be a solution of equation \((1)\) which for some \( t_0 > a \) and \( b \in (a, \infty) \) is oscillatory on \([t_0, b)\). Then \( u \) is unbounded on \([t_0, b)\).

Proof: It again follows from the relations \((6)\). If we suppose that \( u \) is bounded on \([t_0, b)\), then from the third relation in \((6)\) and from the Cauchy Criterion we obtain that \( u \) can be extended to \( b \), too.
The paper [5] contains a theorem of I. Ličko and M. Švec that we restate for the equation (1), $\alpha > 1$ and odd.

**Theorem B:** A necessary and sufficient condition for either oscillatory or monotonic convergence to zero together with its first and second derivative of each solution of the equation (1) on $[t_0, \infty)$ ($t_0 > a$) is that $p(t) > 0$ for all $t > a$ and $\int_{t_0}^{\infty} t^2 p(t) \, dt = \infty$.

The problem is which solutions of equation (1) are oscillatory on the interval $(t_0, \infty)$ and which on the subinterval $\mathcal{I} \subset (a, \infty)$.

**Theorem 5:** Suppose that $p$ fulfills the conditions of Theorem B. Then each solution $u$ of equation (1) defined on the subinterval $\mathcal{I} \subset (a, \infty)$ and such that

$$u(t_0)u''(t_0) - u''(t_0)/2 = -\delta < 0 \text{ for some } t_0 \in \mathcal{I}$$

(10)

is oscillatory for $t > t_0$.

**Proof:** Let $t_0 \in (a, \infty)$ and let $u$ be a solution of equation (1) with the property (10) and defined on $\mathcal{I}$. Then either $\mathcal{I}$ is bounded from above or $\mathcal{I} = [t_0, \infty)$. In the first case $u$ must be oscillatory for $t > t_0$ as follows from Lemma 2. In the second case let us suppose that $u(t) > 0$ for $t \in (t_0, \infty)$. Theorem B then implies that $u'(t) < 0$, $u''(t) > 0$ for all $t > t_0$, for some $t_0 \geq t_0$, and $u(t) \to 0$, $u'(t) \to 0$ as $t \to \infty$. However from the integral identity (4) we get $u(t)u''(t) - u''(t)/2 + \int_{t_0}^{t} p(t)u''(t) \, dt = -\delta < 0$, which implies

$$u''(t)/2 = u(t)u''(t) + \int_{t_0}^{t} p(t)u''(t) \, dt + \delta \geq \delta \text{ for all } t \geq t_0,$$

but this contradicts the assumption $u'(t) \to 0$ as $t \to \infty$.

**Theorem 6:** Suppose that $p$ satisfies the conditions of Theorem B. Then each solution $u$ of equation (1) with double null point at $t_0 > a$ oscillates on the right of $t_0$.

**Proof:** Again there are two cases. In the case when $u$ is defined on a bounded from above interval it must, by Lemma 2, oscillate. In the second case when $u$ is defined on $[t_0, \infty)$ and we suppose that $u(t) > 0$ for all $t > t_1$, for some $t_1 \geq t_0$, it has to converge together with its first and second derivatives to zero as $t \to \infty$ and moreover it has to satisfy $u(t) > 0$, $u'(t) < 0$, $u''(t) > 0$ for all $t > t_2$, for some $t_2 \geq t_1$. Let us substitute $u$ into equation (2) and suppose that $v$ is its solution with the property $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$. From Corollary 1 we have that $v(t) > 0$, $v'(t) > 0$, $v''(t) > 0$ for all $t > t_0$. We use $u$ and $v$ to generate equation (3), i.e.

$$v'u''' - v'u' + v'u = 0.$$  

(11)

If $u$ is non-oscillatory we get from equation (11) a contradiction from the fact that $v(t)u''(t) - v'(t)u'(t) + v''(t)u(t) > 0$ for all $t > t_2$.

Let $p(t) > 0$ for all $t \in (a, \infty)$ and let $u$ be a solution of equation (1) defined on $\mathcal{I}$ and satisfying $u(t_0) = u'(t_0) = 0$, $u''(t_0) > 0$ for some $t_0 \in \mathcal{I}$. Further let $v$ be a solution of equation (2) defined on $\mathcal{I}$ and satisfying $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$. Then equation (11) holds for $t > t_0$, where $v(t) > 0$, $v'(t) > 0$ for $t > t_0$. Let us make the substitution $u = \sqrt{v} \, y$ into equation (11). It then takes the form

$$y'' + (3v''/2v - 3v'^2/4v^2)y = 0.$$  

(12)
From the integral identity (5) for \( v \) and \( t > t_0 \) we get

\[
3v''(t)/2v(t) - 3v^2(t)/4v^2(t) = 3/2v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau
\]

and equation (12) is transformed into

\[
v''(t) + \left(3/2v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau\right)v = 0.
\]  

(13)

From the reasoning above we obtain

**Theorem 7:** A necessary and sufficient condition for a solution \( u \) of equation (1) to be oscillatory for \( t > t_0 \in \mathfrak{T} \) is that equation (13) or (12) is oscillatory for \( t > t_0 \).

Apparently, Theorem 7 does not have any practical significance for determination of oscillatoricity or non-oscillatoricity of solutions of equation (1). However, as we shall see in the following, it has a theoretical importance.

**Corollary 3:** Let \( p(t) > 0 \) for \( t \in (a, \infty) \) and let \( u \) be a non-extendable solution of equation (1) on \( (t_0, b) \), \( a < t_0 < b < \infty \), with the property \( u(t_0) = 0 \), \( u'(t_0) \geq 0 \), \( u''(t_0) > 0 \). Then

\[
\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau \rightarrow \infty \text{ as } t \rightarrow b.
\]

**Proof:** From Lemma 3 we have that \( u \) is oscillatory on \( (t_0, b) \) and from equation (1) it follows that the limit of its null points is \( b \). Suppose that \( \nu \) is a solution of equation (2) which is adjoint to the solution \( u \) and has the property \( \nu(t_0) = \nu'(t_0) = 0 \), \( \nu''(t_0) > 0 \). From Corollary 1 we have that \( \nu(t) > 0 \), \( \nu'(t) > 0 \), \( \nu''(t) > 0 \) for all \( t > t_0 \). The function \( u \) is obviously a solution of equation (11) and hence by Theorem (7) equation (13) must be oscillatory on \( (t_0, b) \), \( b < \infty \). This is possible only if

\[
1/v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau \rightarrow \infty \text{ as } t \rightarrow b.
\]

(14)

However for \( t > t_0 \) clearly the inequality

\[
1/v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau \leq \int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}\,d\tau
\]

holds. Hence the assertion follows.

**Corollary 4:** Suppose that the assumptions of Corollary 3 hold. Then any solution \( v \) of the equation (2) satisfying the condition \( v(t_0) = v'(t_0) = 0 \), \( v''(t_0) > 0 \) has the property \( v(t) \rightarrow \infty \), \( v'(t) \rightarrow \infty \), \( v''(t) \rightarrow \infty \) as \( t \rightarrow b \).

**Proof:** Relation (14) implies the relation

\[
\int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau \rightarrow \infty \text{ as } t \rightarrow b.
\]

(15)

The integral identity (5) for the solution \( v \) has the form

\[
v(t)v''(t) - v^2(t)\nu/2 - \int_{t_0}^{t} p(\tau)u^{\alpha - \xi(\tau)}v^2(\tau)\,d\tau = 0.
\]

Suppose that \( v \) is bounded on \( (t_0, b) \). Then
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\[ v(t) v''(t) = v' (t)^2 / 2 + \int_{t_0}^{t} p(t) u(t) v^2(t) \, dt. \]

From this and from relation (15) it follows that \( v''(t) \to \infty \) as \( t \to b \) and therefore also \( v'(t) \to \infty \), \( v(t) \to \infty \) as \( t \to \infty \). But this is in contradiction with the assumption that \( v \) is bounded.

Suppose we have linear differential equations of the third order

\[ (p_1) \quad y''' + p_1 y = 0 \quad \text{and} \quad (p_2) \quad z''' + p_2 z = 0, \]

where \( p_1, p_2 \) are continuous functions on \((a, \infty)\), \( p_1(t) > 0 \) and \( p_2(t) > 0 \) for all \( t \in (a, \infty) \).

**Lemma 4:** Let \( p_1 \leq p_2 \) on \((a, \infty)\). If equation \((p_2)\) is non-oscillatory in \((a, \infty)\) (i.e. each of its solutions has at most a finite number of null points in \((a, \infty)\)), then the equation \((p_1)\) is also non-oscillatory in \((a, \infty)\).

**Proof:** The assertion is contained in Theorem 2.5 and Corollary 2.5 of [2], respectively.

Let us denote the adjoint equation \( z''' - p_1 z = 0 \) to equation \((p_1)\) by \((\bar{p_1})\).

**Lemma 5:** Let \( p_1(t) > 0 \) for \( t \in (a, \infty) \) and \( w \) be a solution of equation \((\bar{p_1})\) with the property \( w(t_0) = w'(t_0) = 0 \), \( w'(t_0) > 0 \) for some \( t_0 \in (a, \infty) \). Then the set of solutions \( y \) of equation \((p_1)\) with the property \( y(t_0) = 0 \) (called the bundle of solutions of equation \((p_1)\) in the point \( t_0 \)) satisfies the equation \((w)\)

\[ w''' - w'y + w'y = 0. \]

Differentiating equation \((w)\) term by term we obtain the equation \((p_1)\). If equation \((w)\) is non-oscillatory on \((t_0, \infty)\), then equation \((p_1)\) is also non-oscillatory on \((t_0, \infty)\).

The proof of this lemma is not included since it is the basic property of linear equations of third order [2].

**Remark 2:** The assertion of Lemma 5 holds for arbitrary solutions \( w \) of equation \((\bar{p_1})\), but the interesting case is \( w(t) \neq 0 \) for \( t > t_0 \).

**Theorem 8:** Suppose \( p(t) > 0 \) for all \( t \in (a, \infty) \) and let \( f \) be a given function with continuous third derivative, \( f(t) > 0 \) and \( f'''(t) > 0 \) for all \( t \in (a, \infty) \), such that the equation

\[ y''' + (3f''/2f - 3f'^3/4f^2)y = 0 \quad (16) \]

is non-oscillatory in \((a, \infty)\). Then each solution \( \bar{u} \) of equation \((1)\), with the property \( \bar{u}(t_0) = 0 \) for some \( t_0 > a \) and which is defined on \((t_0, \infty)\) and satisfies the inequality

\[ p(t) \bar{u}'(t)/(f(t)) \leq f'''(t)/(f(t)) \quad \text{for all} \; t \geq t_0, \quad (17) \]

is non-oscillatory on \((t_0, \infty)\).

**Proof:** Besides of the equation

\[ u''' + p \bar{u}' u = 0 \quad (18) \]

we have the equation

\[ v''' + (f'''/f)v = 0, \quad (19) \]

that has been obtained by differentiating the equation.
\[ f'v'' - f'v' + f''v = 0 \]  \hspace{1cm} (20)

and which by the transformation \( v = \sqrt{T}y \) can be converted into the equation (16). From the assumption that equation (16) is non-oscillatory on \( \langle t_0, \infty \rangle \) it follows that equation (20) is non-oscillatory on \( \langle t_0, \infty \rangle \) and from Lemma 5 we have that equation (19) is non-oscillatory, too. From the assumption (17) and from Lemma 4 it follows that equation (18) is non-oscillatory on \( \langle t_0, \infty \rangle \). Since \( \overline{u} \) is a solution of equation (18), it is therefore non-oscillatory on \( \langle t_0, \infty \rangle \).  

**Corollary 5:** Let \( f(t) = t^n \), where \( n = 1 + 2/\sqrt{3} \) and let \( a > 0 \). Then the equation (1) does not have an oscillatory solution \( \overline{u} \) with null point in the point \( t_0 > a \) on the interval \( \langle t_0, \infty \rangle \) that would satisfy the relation (17), i.e. the relation

\[ \overline{u}^{\alpha} - \overline{u}(t) \leq 2/(3\sqrt{3}t^3p(t)) \text{ for all } t \geq t_0. \]  \hspace{1cm} (21)

**Proof:** The equation (16) has the form

\[ y'' + (3(n^2 - 2n)/4t^2)y = 0. \]  \hspace{1cm} (22)

Let \( 3(n^2 - 2n) = 1 \). The positive root of this equation is \( n = 1 + 2/\sqrt{3} \). By the well-known Kneser criterion equation (22) is non-oscillatory and hence equation (19) is non-oscillatory if \( f''(t)/f(t) = n(n - 1)(n - 2)/t^2 = 2/(3\sqrt{3}t^3) \). This and the relation (17) imply the relation (21).

**REFERENCES**


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