On Uryson Operators with Partial Integrals in Lebesgue Spaces with Mixed Norm

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Abstract. In this paper we consider some class of partial Uryson integral operators in spaces with mixed norm. We give some conditions for action, boundedness and continuity of those operators.

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1. Introduction

Let $T \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{R}^m$ be two compact sets with Lebesgue measure, $D = T \times S$, $a_i : D \times \mathbb{R} \rightarrow \mathbb{R}$ $(i = 1, 2, 3)$ given Carathéodory functions, and

$$A = A_1 + A_2 + A_3,$$

where

$$\begin{align*}
(A_1 x)(t, s) &= \int_T a_1(t, s, x(\tau, s)) \, d\tau \\
(A_2 x)(t, s) &= \int_S a_2(t, s, x(t, \sigma)) \, d\sigma \\
(A_3 x)(t, s) &= \iint_D a_3(t, s, x(\tau, \sigma)) \, d\tau d\sigma.
\end{align*}$$

The operators $A$, $A_1$, and $A_2$ are so called partial Uryson integral operators, which have been studied in $C(D)$, in spaces with mixed quasinorm $L^0[L^\beta]$, and in quasi-Banach ideal spaces (see [1, 4, 8], respectively). The properties of partial Uryson integral operators essentially differ from those of ordinary Uryson integral operators. For example, the operator $A_1$ with kernel $a_1(t, s, u) \equiv u$ is not completely continuous in $L^p(D)$, but the operator $A_3$ is completely continuous for $a_3(t, s, u) \equiv u$.

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We remark that linear and nonlinear operators with partial integrals have applications in problems of continuum mechanics, of the theory of transfer, of differential and integro-differential equations and other questions.

In this paper the operator $A$ is studied in Lebesgue spaces with mixed norm

$$L^q(T)[L^p(S)] \quad (1 \leq p, q \leq \infty).$$

More general classes of partial Uryson integral operators in spaces of summable functions have been studied in [3]. Action, boundedness, and continuity criteria of the operator $A_3$ in Lebesgue spaces have been obtained by Ojnarov [6].

2. Action, boundedness and continuity

Let $M(D)$ be the space of all real measurable almost everywhere finite functions on $D$. We denote by $L^q(T)[L^p(S)] \quad (1 \leq p, q \leq \infty)$ the set of functions $x \in M(D)$ for which the norm

$$\|x\|_{L^q(T)[L^p(S)]} = \|t \mapsto \|x(t, \cdot)\|_{L^p(S)}\|_{L^q(T)}$$

is defined and finite [2]. These spaces are Banach spaces. Of course, in case $p = q$ we have $L^p(T)[L^p(S)] \cong L^p(T \times S)$. Let $X(x_0, r)$ denote the closed ball of radius $r$ with center $x_0$ in the space $X$.

The operators $A_1$, $A_2$, and $A_3$ are defined on functions $x \in M(D)$ for which the functions

$$(t, s, \tau) \mapsto a_1(t, s, x(\tau, s))$$

$$(t, s, \sigma) \mapsto a_2(t, s, x(t, \sigma))$$

$$(t, s, \tau, \sigma) \mapsto a_3(t, s, x(\tau, \sigma))$$

are summable in the variables $\tau$, $\sigma$, and $(\tau, \sigma)$, respectively, for almost all $(t, s) \in D$. Let $D(A)$ be the domain of definition of the operator $A = A_1 + A_2 + A_3$. If $a_1(t, s, 0) = a_2(t, s, 0) = a_3(t, s, 0) = 0$, and $x_1, \ldots, x_n \in D(A)$ are functions with disjoint supports, then $x_1 + \ldots + x_n \in D(A)$ and the operator $A$ is partially additive, i.e.,

$$A(x_1 + \ldots + x_n) = Ax_1 + \ldots + Ax_n.$$ 

In general, the operator $x \mapsto A(x + x_0) - Ax_0$ is partially additive for fixed $x_0 \in D(A)$.

By the partial additivity of $A$ we have the following statement.

**Theorem 1.** Let $X = L^q(T)[L^p(S)] \quad (1 \leq p, q < \infty)$. Suppose that the operator $A$ acts from $X(x_0, r)$ into $Y = L^\beta(T)[L^\alpha(S)] \quad (1 \leq \alpha, \beta \leq \infty)$. Then $A$ acts from $X$ into $Y$ and is bounded (i.e., $A$ is bounded on any bounded set). Moreover, $A$ is continuous on $X$ if $A$ is continuous on $X(x_0, r)$.

By Theorem 1 the boundedness of the operator $A$ follows direct from its action.

The next theorem concerning acting conditions (both sufficient and necessary) for the operator $A_3$ may be obtained following the idea of [6].
Theorem 2. The operator $A_3$ acts from $X = L^p(T)[L^q(S)]$ $(1 \leq p, q < \infty)$ into $Y = L^\alpha(T)[L^\beta(S)]$ $(1 \leq \alpha, \beta \leq \infty)$ if and only if, for any $u \in \mathbb{R}$,
\[
\|a_3(\cdot, \cdot, u)\|_Y \leq a |u|^{\min\{p, q\}} + b,
\]
where $a$ and $b$ are non-negative constants.

Proof. Without loss of generality, we assume that $\text{mes}T = \text{mes}S = 1$. Suppose that condition (4) holds. Then for any $x \in X$ the Hölder and Minkowski inequalities imply that
\[
\|A_3 x\|_Y = \left\| \int_D a_3(\cdot, \cdot, x(\tau, \sigma)) d\tau d\sigma \right\|_Y \\
\leq \int_D \left\| a_3(\cdot, \cdot, x(\tau, \sigma)) \right\|_Y d\tau d\sigma \\
\leq \int_D (a|x(\tau, \sigma)|^{\min\{p, q\}} + b) d\tau d\sigma \\
\leq a \|x\|_X^{\min\{p, q\}} + b.
\]
Hence, $A_3$ acts from $X$ into $Y$.

Conversely, suppose that the operator $A_3$ acts from $X$ into $Y$. Then, by Theorem 1, there exists a number $b > 0$ such that $\|A_3 x\|_Y \leq b$ if $\|x\|_X \leq 1$. Let $u \in \mathbb{R}$ and $x_u \equiv u \in X$. If $|u| \leq 1$, it is clear that
\[
\|a_3(\cdot, \cdot, u)\|_Y = \|A_3 x_u\|_Y \leq b \leq b(|u|^{\min\{p, q\}} + 1).
\]
If $|u| > 1$, we define a function $\tilde{x}_u$ on $D$ by
\[
\tilde{x}_u(t, s) = \begin{cases} 
  u\chi_{T \times S_u}(t, s) & \text{if } p \leq q \\
  u\chi_{T_u \times S}(t, s) & \text{if } p > q
\end{cases}
\]
where $S_u$ is a measurable subset of $S$ with $\text{mes}S_u = |u|^{-p}$ and $T_u$ is a measurable subset of $T$ with $\text{mes}T_u = |u|^{-q}$. Here, $\chi_{T \times S_u}$ and $\chi_{T_u \times S}$ denote the characteristic functions of $T \times S_u$ and $T_u \times S$, respectively. Then $\|\tilde{x}_u\|_X = 1$ and
\[
\left\| |u|^{-\min\{p, q\}} a_3(\cdot, \cdot, u) + (1 - |u|^{-\min\{p, q\}}) a_3(\cdot, \cdot, 0) \right\|_Y = \|A_3 \tilde{x}_u\|_Y \leq b.
\]
Hence,
\[
\|a_3(\cdot, \cdot, u)\|_Y \leq b|u|^{\min\{p, q\}} + |u|^{\min\{p, q\}}\|a_3(\cdot, \cdot, 0)\|_Y \leq 2b|u|^{\min\{p, q\}}.
\]
From (5) and (6) it follows that condition (4) holds.

Some acting conditions for the operators $A_1$ and $A_2$ in spaces of summable functions have been given in [3]. We will give simple acting conditions (only sufficient) in the next lemma.
Lemma. Let \(1 \leq p, q, \alpha, \beta < \infty\), \(X = L^q(T)[L^p(S)]\), and \(Y = L^\beta(T)[L^\alpha(S)]\). The operators \(A_1\) and \(A_2\) act from \(X\) into \(Y\) if the kernels \(a_1\) and \(a_2\) satisfy growth conditions of the form

\[
|a_i(t, s, u)| \leq c_1 |u|^{\min\{p, q\}/\rho_i(\alpha, \beta)} + b_i(t, s) \quad (i = 1, 2)
\]  

(7)

for some \(b_1, b_2 \in Y\) and \(c_1, c_2 \geq 0\), where \(\rho_1(\alpha, \beta) = \alpha\) and \(\rho_2(\alpha, \beta) = \beta\). Moreover, in this case \(A_1\) and \(A_2\) are bounded and continuous.

Proof. It is easy to show the first statement by the Hölder and Minkowski inequalities. The continuity of \(A_1\) and \(A_2\) follows from the principle of majorants [8].

We note that the growth condition (7) is not necessary for the action of \(A_1\) (resp. \(A_2\)). Moreover, there exists \(A_1\) acting from \(X\) into \(Y\) (whence \(A_1\) is even bounded), which is not continuous. In particular, the corresponding kernel \(a_1\) does not satisfy the growth condition (7) (by the previous lemma).

The following example is essentially due to P. P. Zabrejko [5].

Example. Let \(D = [0, 1] \times [0, 1]\), \(X = L^q(T)[L^p(S)]\), and \(Y = L^\beta(D)\) \((1 \leq p, q, \beta < \infty)\). Let \(z_n(t, s) = z_n(t) \geq 0\) have disjoint support, and \(|z_n|_Y = 1\). Define the kernel \(a_1\) on \(D \times \mathbb{R}\) by

\[
a_1(t, s, u) = \begin{cases} 
(2^n|u| - 1)z_{n-1}(t) + (2 - 2^n|u|)z_n(t) & \text{if } 2^{-n} \leq |u| < 2^{1-n} \\
0 & \text{if } u = 0 \text{ or } |u| \geq 1.
\end{cases}
\]

Then the kernel \(a_1\) is a non-negative Carathéodory function, and the operator \(A_1\) acts from \(X\) into \(Y\) and is bounded (it even has bounded range): Indeed, by Minkowski’s inequality we have for any measurable \(x\)

\[
\|A_1 x\|_Y^\beta = \int_0^1 \left\| \int_0^1 a_1(\cdot, s, x(\tau, s)) d\tau \right\|_{L^\beta}^\beta ds \leq \int_0^1 \left( \int_0^1 \left\| a_1(\cdot, s, x(\tau, s)) \right\|_{L^\beta} d\tau \right)^\beta ds \leq 1.
\]

However, \(A_1\) is not continuous, since it maps the convergent sequence \((z_n) = (2^{-n})\) into the non-compact sequence \((A_1 x_n) = (z_n)\).

The kernel \(a_1\) not only fails to satisfy the growth condition (7). Even more, \(a_1\) does not satisfy

\[
|a_1(t, s, u)| \leq c |u|^\gamma + b(t, s) \quad (8)
\]

for fixed \(c, \gamma > 0\) and \(b \in Y\). Indeed, for \(u_n = 2^{-n}\), (8) would imply \(z_n(t, s) = a_1(t, s, u_n) \leq c + b(t, s)\), whence \(d(t, s) = b(t, s) + c\) satisfies \(d \geq z_n\) for all \(n\), which obviously is not possible, since \(d \in Y\).
The continuity of the operator $A_3$ does not follow from its action and boundedness as is shown by the previous example (consider $a_3 = a_1$).

To discuss continuity conditions for the operator $A_3$, we apply the following theorem. Recall that a set $G \subset X$ is absolutely bounded if $\sup\{\|\chi_G x\|_X : x \in G\} \to 0$ as $\mes\Omega \to 0$.

**Theorem 3.** Let $1 \leq p, q, \alpha, \beta < \infty$, $X = L^q(T)[L^p(S)]$ and $Y = L^\beta(T)[L^\alpha(S)]$. Suppose that, for each function $x \in X$,

$$\left\| \int_D |a_3(\cdot, \cdot, x(\tau, \sigma))|d\tau d\sigma \right\|_Y < \infty.$$ 

Then the operator $A_3$ acts from $X$ into $Y$. Moreover, for each absolutely bounded set $G \subset X$ and for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality

$$\sup_{x \in G} \left\| \int_{D_1} |a_3(\cdot, \cdot, x(\tau, \sigma))|d\tau d\sigma \right\|_Y < \varepsilon$$ 

holds whenever $D_1 \subset D$ satisfies $\mes D_1 < \delta$.

**Proof.** It is analogous to that of [5: Theorem 18.4].

**Theorem 4** (see [7]). Let $1 \leq p, q, \alpha, \beta < \infty$. The operator $A_3$ acts from $X = L^q(T)[L^p(S)]$ into $Y = L^\beta(T)[L^\alpha(S)]$ and is continuous if and only if condition (4) holds and

$$\lim_{u \to u_0} \left\| a_3(\cdot, \cdot, u) - a_3(\cdot, \cdot, u_0) \right\|_Y = 0$$

(9)

for any $u_0 \in \mathbb{R}$.

**Proof.** Without loss of generality, assume $a_3(t, s, 0) = 0$ and $\mes T = \mes S = 1$. Suppose that the operator $A_3$ acts from $X$ into $Y$ and is continuous. Then condition (4) holds by Theorem 2. Putting $x \equiv u$ and $x_0 \equiv u_0$, we have $A_3 x = a_3(\cdot, \cdot, u)$ and $A_3 x_0 = a_3(\cdot, \cdot, u_0)$. Thus the continuity of $A_3$ implies (9).

Conversely, suppose that conditions (4) and (9) hold. Then the operator $A_3$ acts from $X$ into $Y$ by Theorem 2. Assume that $A_3$ is not continuous. This means that there exist a sequence $(x_n)$ converging to a function $x_0$ in $X$ and a number $\varepsilon_0 > 0$ such that

$$\|A_3 x_n - A_3 x_0\|_Y \geq \varepsilon_0 \quad (n \in \mathbb{N}).$$

(10)

Since $x_n \to x_0$ in $X$, the set $\{x_0, x_1, x_2, \ldots\}$ is absolutely bounded. Hence, by Theorem 3 there is a number $\delta > 0$ such that the inequalities

$$\|A_3 F x_n\|_Y < \frac{\varepsilon_0}{3} \quad (n \geq 0)$$

(11)

hold whenever $F \subset D$ satisfies $\mes F < \delta$. Let $c = \sup_{n \geq 0} \|x_n\|_X$, $N = c(\frac{\varepsilon}{3})^{-1/\min\{p, q\}}$, and $D_n^N = \{ (t, s) : |x_n(t, s)| \geq N \} \quad (n \geq 0)$. Then $\mes D_n^N \leq \frac{\varepsilon}{3} \quad (n \geq 0)$. Since $x_n \to x_0$ in $X$, we can find a subsequence $(x_{n_k})$ which converges almost everywhere to $x_0$. Moreover, by Egorov’s theorem, there exists a measurable set $D_\delta \subset D$ such that
\[ \text{mes}(D - D_\delta) < \frac{\delta}{3} \quad \text{and} \quad (x_{n_k}) \text{ converges to } x_0 \text{ uniformly on } D_\delta. \]

Let \( F_k^\delta = D_\delta - (D_n \cup D_0^N) \) and \( \tilde{F}_k^\delta = D - F_k^\delta \) \((k \geq 1)\). Then \( \text{mes} \tilde{F}_k^\delta < \delta \) for any \( k \geq 1 \). Now, we estimate

\[
\| A_3x_{n_k} - A_3x_0 \|_Y \leq \| A_3\chi_{\tilde{F}_k^\delta}x_{n_k} - A_3\chi_{\tilde{F}_k^\delta}x_0 \|_Y + \| A_3\chi_{\tilde{F}_k^\delta}x_{n_k} \|_Y + \| A_3\chi_{\tilde{F}_k^\delta}x_0 \|_Y. \]

By condition (9) there is a \( \delta_0 = \delta_0(N, \varepsilon_0) > 0 \) such that

\[
\| a_3(\cdot, \cdot, u) - a_3(\cdot, \cdot, u_0) \|_Y < \frac{\varepsilon_0}{3} \quad (12)
\]

whenever \(|u| < N, |u_0| < N, \) and \(|u - u_0| < \delta_0 \). Since \((x_{n_k})\) converges to \( x_0 \) uniformly on \( D_\delta \), there exists an integer \( m = m(\delta_0) \) such that \(|x_{nm}(t, s) - x_0(t, s)| < \delta_0 \) for all \((t, s) \in D_\delta\). Combining inequalities (11) - (13) we get \( \| A_3x_{n_m} - A_3x_0 \|_Y < \varepsilon_0 \), which is contradictory to (10). Thus the operator \( A_3 \) is continuous.

References


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