On the $K$-Theory Groups of Certain Crossed Product $C^*$-Algebras

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Abstract. Given $\alpha, \beta$ two irrational numbers in $(0,1)$ we define an analogue of the irrational rotation $C^*$-algebra on $T^2 = S^1 \times S^1$. Hence we compute explicitly the $K$-theory groups associated to that crossed product $C^*$-algebra.

Keywords: $C^*$-algebras, crossed product $C^*$-algebras, $K$-theory

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Let $C(T^2)$ be the $C^*$-algebra of continuous functions on $T^2 = S^1 \times S^1$ and let $\rho(\alpha, \beta) : T^2 \rightarrow T^2$ be the automorphism corresponding to a rotation of angles $2\pi \alpha, 2\pi \beta$ where $\alpha, \beta$ are irrational numbers, i.e., $(z_1, z_2) \mapsto (e^{2\pi i \alpha} z_1, e^{2\pi i \beta} z_2)$. Then the group $\mathbb{Z}$ acts as a transformation group on

$$T^2 : (z_1, z_2) \mapsto (e^{2\pi i n \alpha} z_1, e^{2\pi i n \beta} z_2)$$

by powers of the rotation automorphism. We can extend this action to the $C^*$-algebra of continuous functions on $T^2$ so it gives an action of $\mathbb{Z}$ on $C(T^2)$. We define $A(\alpha, \beta)$ to be the crossed product $C^*$-algebra of $C(T^2)$ under the automorphism $\rho(\alpha, \beta) : n \mapsto \rho(\alpha, \beta)(n) \,(n \in \mathbb{Z})$. By [4] we have that $A(\alpha, \beta) = C(T^2) \times \rho(\alpha, \beta) \mathbb{Z}$ is a simple $C^*$-algebra.

Denote by $U_1$ the unitary operator in $A(\alpha, \beta)$ corresponding to the element $n = 1 \in \mathbb{Z}$ and defined by

$$(U_1 f)(z_1, z_2) = f((e^{2\pi i \alpha} z_1, z_2)) \quad (f \in L^2(T^2))$$

and similarly denote

$$(V_1 f)(z_1, z_2) = f(z_1, (e^{2\pi i \beta} z_2)) \quad (f \in L^2(T^2)).$$

Let $W_1$ be the multiplication operator on $L^2(T^2)$. Therefore we have

$$U_1 V_1 = V_1 U_1, \quad U_1 W_1 = (e^{2\pi i \alpha}) W_1 U_1, \quad V_1 W_1 = (e^{2\pi i \beta}) W_1 V_1. \quad (1)$$

Let $C^*(U, V, W)$ be the $C^*$-algebra generated by three unitary operators $U, V, W$ such that they satisfy relations (1). Then by [3: Formula (7.6.6)] and by the fact that $A(\alpha, \beta)$

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is simple we can conclude that $A_{(\alpha,\beta)}$ is isomorphic to the $C^*$-algebra $C^*(U,V,W)$, where the isomorphism maps $U_1 \rightarrow U, V_1 \rightarrow V, W_1 \rightarrow W$. Let $A_\alpha = C(T) \times \rho_\alpha Z$ be the irrotational rotation $C^*$-algebra, i.e., the crossed product of the $C^*$-algebra of the continuous functions on the unit circle by the automorphism corresponding to a rotation of angle $2\pi \alpha$. This $C^*$-algebra is simple (see [4]). It is generated by two unitaries $u, w$ such that $u w = (e^{2\pi i \alpha}) w u$ (see [5]).

By the same argument we can conclude that $A_\alpha$ is contained in $A_{(\alpha,\beta)}$.

Let $\gamma_\beta$ be the automorphism of $A_\alpha$ defined by $\gamma_\beta = a d V$, with $V$ the unitary operator in $A_{(\alpha,\beta)}$. Then we form the $C^*$-algebra crossed product $A_\alpha \times_{\gamma_\beta} Z$. Let $\gamma_\beta(u) = V u V^* = u \quad \text{and} \quad \gamma_\beta(w) = V w V^* = (e^{-2\pi i \beta}) w$

so that the automorphism $\gamma_\beta$ leaves the operator $u$ fixed. Therefore we have $u V = V u, u w = (e^{2\pi i \alpha}) w u$ and $V w = (e^{2\pi i \beta}) w V$. By using [1: Prop. 5.1] we have $\Gamma(\alpha) = T$ and by [2: Formula (8.11.12)] we get that the $C^*$-algebra $A_\alpha \times_{\gamma_\beta} Z$ is simple.

Similarly we can consider the irrational rotation $C^*$-algebra $A_\beta$ generated by $v, w$ and form the crossed product $C^*$-algebra $A_\beta \times_{\gamma_\beta} Z$ under the automorphism $\gamma_\beta$ given by $ad U$. Let $(\pi, V)$ be a covariant representation of $(A_\alpha, \gamma_\beta Z)$. Let $\tilde{\pi} : A_\alpha \times_{\gamma_\beta} Z \rightarrow A_{(\alpha,\beta)}$ be defined by $u \rightarrow U_1, w \rightarrow W_1, v \rightarrow V_1$. Then by [3: Formula (7.6.6)], since both $C^*$-algebras are simple $\tilde{\pi}$ gives an isomorphism. Similarly $A_{(\alpha,\beta)}$ and $A_\beta \times_{\gamma_\alpha} Z$ are isomorphic. Therefore the $C^*$-algebras $A_{(\alpha,\beta)}, A_\alpha \times_{\gamma_\beta} Z$ and $A_\beta \times_{\gamma_\alpha} Z$ are all isomorphic.

Let $A_\alpha = C(T) \times \rho_\alpha Z$ and $A_\beta \times_{\gamma_\alpha} Z$ be the irrational rotation $C^*$-algebras and let $p$ and $\bar{p}$ be the Rieffel projections in $A_\alpha$ and $A_\beta$, respectively (see [5]). There exists a unitary operator $U$ in $M_n(A_{(\alpha,\beta)})$ such that for the boundary map $\partial$ we have $\partial[U] = [p]$, where $M_n$ stands for the $n \times n$-matrices. As in [6: Section 8.5], let us call $[Q]$ the Bott generator of $K_0(C(T^2))$.

Our main result is the following

**Theorem.** The following two statements are true.

(a) The group $K_0(C(T^2) \times_{\rho(\alpha,\beta)} Z)$ is isomorphic to $Z^4$ and it is generated by $[1], [p], [\bar{p}], [Q]$.

(b) The group $K_1(C(T^2) \times_{\rho(\alpha,\beta)} Z)$ is isomorphic to $Z^4$ and it is generated by $[U], [V], [W], [\bar{U}]$.

**Proof.** (a) Let $C(T^2) \times_{\rho(\alpha,\beta)} Z$ be the $C^*$-algebra generated by three unitary operators $U, V, W$ satisfying relations (1). Let the $C^*$-algebra $C^*(U, V)$ be generated by $U, V, C^*(U, W)$ by $U, W$ and $C^*(V, W)$ by $V, W$. Observe the identities

$$C^*(U, V) = C(T^2), \quad C^*(U, W) = A_\alpha, \quad C^*(V, W) = A_\beta$$

and that all of these $C^*$-algebras are contained in $A_{(\alpha,\beta)}$. By [3] we have the following
exact sequence:

\[ K_0(A_0) \xrightarrow{(id)^* - (\gamma_0)^*} K_0(A_0) \xrightarrow{i^*} K_0(A_0 \times \mathbb{Z}_{\gamma_0}) \]

\[ \delta \]

\[ K_1(A_0 \times \mathbb{Z}_{\gamma_0}) \xleftarrow{i^*} K_1(A_0) \xrightarrow{(id)^* - (\gamma_0)^*} K_1(A_0) \]

(2)

where \( i : A_0 \to A_0 \times \gamma_0 \mathbb{Z} \) is the inclusion map and \( \delta \) is the index map coming from the Toeplitz extension for \((A_0, \gamma_0)\). Starting from the left upper side of the sequence we see that, since \( K_0(A_0) \) is generated by \([1]\) and \([p]\), \((id)^* - (\gamma_0)^*\) is the zero map on \( K_0(A_0) \), then \([1]\) and \([p]\) generate a copy of \( \mathbb{Z}^2 \) in \( K_0(A_0 \times \gamma_0 \mathbb{Z}) \). On \( K_1(A_0) \) the map \((id)^* - (\gamma_0)^*\) is the zero map and because of \( K_1(A_0) \) is generated by \([U]\), \([W]\) we have a copy of \( \mathbb{Z}^2 \) in \( K_1(A_0 \times \gamma_0 \mathbb{Z}) \). Note that since

\[ A_\beta \to A_\beta \times \gamma_0 \mathbb{Z} \quad \text{and} \quad K_0(A_\alpha \times \gamma_0 \mathbb{Z}) \simeq K_0(A_\beta \times \gamma_0 \mathbb{Z}), \quad C^*(W) \to A_\alpha \]

we have the following diagram:

\[ \begin{array}{cccc}
K_0(A_\alpha \times \mathbb{Z}_{\gamma_0}) & \xrightarrow{\delta} & K_1(A_\alpha) & \xrightarrow{(id)^* - (\gamma_0)^*} K_1(A_\alpha) \\
\uparrow & & \uparrow & \uparrow \\
K_1(A_\beta) & \xrightarrow{\delta} & K_1(C^*(W)) & \xrightarrow{0} K_1(C^*(W))
\end{array} \]

(3)

where the vertical rows are the inclusion maps. The above diagram is obtained by considering \( A_\beta \) as the \( C^* \)-algebra crossed product \( C(T) \times \rho_\beta \mathbb{Z} \) where \( C^*(W) = C(T) \). By the exactness of the bottom line of the diagram we get \( \delta[p] = [W] \). Now the rectangles in the diagram commute so also on the top line we have \( \delta[p] = [W] \). Let us show that \( \delta[Q] = [V] \). So considering \( C^*(U, V) = C(T^2) \), then we have the following diagram:

\[ \begin{array}{cccc}
K_0(A_{\alpha, \beta}) & \xrightarrow{\delta} & K_1(C(T^2)) & \xrightarrow{0} K_1(C(T^2)) \\
\uparrow & & \uparrow & \uparrow \\
K_0(C(T^2)) & \xrightarrow{\delta} & K_1(C^*(V)) & \xrightarrow{0} K_1(C^*(V))
\end{array} \]

(4)

where \( C^*(V) = C(T) \). The bottom part is the Pimsner-Voiculescu six term exact sequence for the \( C^* \)-algebra \( C(T^2) \) by seeing it as the crossed product of \( C^*(V) \) by the trivial action of \( \mathbb{Z} \). As in [6: Section 8.5] let \([Q]\) be the generator of \( K_0(C(T^2)) \). Then we have \( \delta[Q] = [V] \) and \( \delta[1] = 0 \), since the diagram commute this holds also for the
top part so $\partial [Q] = [V]$. Hence we can conclude that $K_0(A_{(\alpha, \beta)}) = \mathbb{Z}^4$ with generators $[1], [p], [\bar{p}], [Q]$.

(b) The map $(id)^* - (\gamma_\beta)^*$ on $K_1(A_\alpha)$ is the zero map and because of $K_1(A_\alpha) = \mathbb{Z}^2$ we have a copy of $\mathbb{Z}^2$ in $K_1(A_{(\alpha, \beta)})$ with generators $[U], [W]$. Also at the $K_0(A_\alpha)$ level the map $(id)^* - (\gamma_\beta)^*$ is the zero map. Thus we get another copy of $\mathbb{Z}^2$ in $K_1(A_{(\alpha, \beta)})$. Let $U$ be the unitary operator in $M_n(A_{(\alpha, \beta)})$ such that $I - [U] = [1]$. As in (3) we consider the diagram

\[
\begin{array}{ccc}
K_1(C^*(W)) & \longrightarrow & K_1(A_\alpha) \\
\downarrow & & \downarrow \\
K_1(C^*(W)) & \longrightarrow & K_1(A_\alpha) \\
\downarrow & & \downarrow \\
K_1(A_\beta) & \longrightarrow & K_1(A_{(\alpha, \beta)}) \\
\downarrow & & \downarrow \delta \\
K_0(C^*(W)) & \longrightarrow & K_0(A_\alpha) \\
\end{array}
\]

By the exactness of the bottom line of the diagram we get $\partial[W] = [1]$. Now the rectangles in the diagram commute. Since this holds also for the top part, we have $\partial[W] = [1]$. Thus statement (b) is proved.

References


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