Double Walsh Series with Coefficients of Bounded Variation\(^1\)

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Convergence properties of double Walsh series are studied whose coefficients form a null sequence of bounded variation. These series converge regularly at all points of \((0, 1) \times (0, 1)\) and converge in the pseudometric of \(L^r\) for all \(r \in (0, 1)\). Sufficient conditions for convergence are also proved which involve the second-order differences of the coefficients.

1. Introduction. We will study the convergence behaviour of double Walsh series of the form

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} w_j(x) w_k(y),
\]

where \(\{a_{jk}\}\) is a null sequence of complex (or real) numbers and \(\{w_j\}\) is the well-known Walsh orthonormal system defined on the interval \(I = [0, 1)\) and considered in the Paley enumeration (see, e.g., [1, p. 60]). Thus, series (1.1) is considered on the unit square \(J^2 = [0, 1) \times [0, 1)\). The pointwise convergence of (1.1) is usually defined in Pringsheim's sense (see, e.g., [6, Vol. 2, Ch. 17]). This means that we form the rectangular partial sums \(s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} w_j(x) w_k(y)\), then let both \(m\) and \(n\) tend to \(\infty\), independently of one another, and assign the limit \(f(x, y)\) (if it exists) to series (1.1) as its sum. Following Hardy [3], series (1.1) is said to be regularly convergent if it converges in Pringsheim's sense, and, in addition, each "row series" of (1.1) (i.e., when we delete \(\sum_{k=0}^{\infty}\) in (1.1) and the summation is done only with respect to \(j\) for each fixed \(k\)) as well as each "column series" converges in the ordinary sense of convergence of single series. The notion of regular convergence was rediscovered in [4], where it was

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defined by the following equivalent condition: the sums
\[ s(Q; z, y) = \sum_{j=m}^{M} \sum_{k=n}^{N} a_{jk} w_j(x) w_k(y) \]  
(1.2)
tend to zero as \( \max (m, n) \to \infty \), independently of the choices of \( M (\geq m) \) and \( N (\geq n) \), where \( Q = \{(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0 : m \leq j \leq M \text{ and } n \leq k \leq N\} \).

2. Main results. We remind the reader that the differences \( \Delta_{pq} \) of a double sequence \( \{a_{jk}\} \) are defined for any non-negative integers \( p \) and \( q \) as follows:
\[ \Delta_{00} a_{jk} = a_{jk}, \quad \Delta_{pq} a_{jk} = \begin{cases} \Delta_{p-1,q} a_{jk} - \Delta_{p-1,q} a_{j+1,k} & \text{if } p \geq 1, \\ \Delta_{p,q-1} a_{jk} - \Delta_{p,q-1} a_{j,k+1} & \text{if } q \geq 1. \end{cases} \]

As is well known, the two right-hand sides coincide if \( \min (p, q) \geq 1 \). We mention that a double induction argument gives
\[ \Delta_{pq} a_{mn} = \sum_{j=0}^{p} \sum_{k=0}^{q} (-1)^{j+k} \binom{p}{j} \binom{q}{k} a_{m+j,n+k}. \]

We will prove convergence results for the cases \( p = q = 1 \) and \( p = q = 2 \).

**Theorem 1:** If a double sequence \( \{a_{jk}\} \) is such that
\[ a_{jk} \to 0 \text{ as } \max (j, k) \to \infty \]  
(2.1)
and
\[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| < \infty, \]  
(2.2)
then
(i) series (1.1) converges regularly to some function \( f = f(x, y) \) for all \( 0 < x, y < 1 \);
(ii) for all \( 0 < r < 1 \),
\[ \|s_{mn} - f\|_r \to 0 \text{ as } \min (m, n) \to \infty, \]  
(2.3)
where \( \|\cdot\|_r \) means the pseudonorm in \( L^r(I^2) \) defined by \( \|g\|_r = \int \int [g(x, y)]^r dx dy \).

If condition (2.2) is satisfied, \( \{a_{jk}\} \) is said to be of bounded variation. We note that an analogous theorem was proved in [5] for double trigonometric series.

**Theorem 2:** If a double sequence \( \{a_{jk}\} \) is such that condition (2.1) is satisfied and
\[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{22} a_{jk}| < \infty, \]  
(2.4)
\[ \sum_{j=0}^{\infty} |\Delta_{20} a_{jk}| \text{ is finite for each } k \text{ and tends to } 0 \text{ as } k \to \infty, \]  
(2.5)
\[ \sum_{k=0}^{\infty} |\Delta_{02} a_{jk}| \text{ is finite for each } j \text{ and tends to } 0 \text{ as } j \to \infty, \]  
(2.6)
then conclusion (i) in Theorem 1, except possibly when \( x \) or \( y \) is a dyadic rational, and conclusion (ii) for all \( 0 < r < 1/2 \) hold true.
3. Auxiliary results. We need the following three lemmas.

Lemma 1: If \( \{a_{jk}\} \) satisfies condition (2.1) and for some \( p, q \geq 1 \),

\[
C_{pq} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |A_{pk}a_{jk}| < \infty, \tag{3.1}
\]

then

\[
\sum_{j=0}^{\infty} |A_{p,q-1}a_{jk}| \leq C_{pq} \quad (k = 0, 1, \ldots), \tag{3.2}
\]

\[
\sum_{j=0}^{\infty} |A_{p,q-1}a_{jk}| \to 0 \quad \text{as} \quad k \to \infty, \tag{3.3}
\]

\[
\sup_{k} \sum_{j=m}^{\infty} |A_{p,q-1}a_{jk}| \to 0 \quad \text{as} \quad m \to \infty. \tag{3.4}
\]

Analogous statements hold true for \( A_{p-1, q}a_{jk} \) under the same conditions (2.1) and (3.1) if the roles of \( j \) and \( k \) are interchanged.

Proof: By (2.1), \( A_{p,q-1}a_{jk} = \sum_{k=k_p}^{\infty} A_{pq}a_{jk} \), whence \( \sum_{j=0}^{\infty} |A_{p,q-1}a_{jk}| \leq \sum_{j=0}^{\infty} \sum_{k=k_p}^{\infty} |A_{pq}a_{jk}| \).

Clearly, (3.1) implies both (3.2) and (3.3). Finally, (3.4) is a consequence of (3.3) (applied for large values of \( k \)) and (3.2) (applied for small values of \( k \)).

Now we consider another double sequence \( \{b_{jk}\} \) of numbers with rectangular partial sums \( B_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} b_{jk} \) \( (m, n = 0, 1, \ldots) \). The next two lemmas can easily be verified by performing double summations by parts.

Lemma 2: For all \( 0 \leq m \leq M \) and \( 0 \leq n \leq N \),

\[
\sum_{j=m}^{M} \sum_{k=n}^{N} b_{jk}a_{jk} = \sum_{j=m}^{M} \sum_{k=n}^{N} B_{jk}A_{11}a_{jk} + \sum_{j=m}^{M} B_{jn}A_{10}a_{j,n+1}
\]

\[
- \sum_{j=m}^{M} B_{j,n-1}A_{10}a_{jn} + \sum_{k=n}^{N} B_{mk}A_{01}a_{M+1,k}
\]

\[
- \sum_{k=n}^{N} B_{m-1,k}A_{01}a_{m,k} + B_{MN}a_{M+1,N+1}.
\]

We introduce the notation

\[
R_{mn} = \{(j, k) \in \mathbb{N}_0 \times \mathbb{N}_0: \text{either } j \geq m + 1 \text{ or } k \geq n + 1\} \tag{3.5}
\]

and let \( \sum \ldots \text{ stand for } \sum_{(j,k) \in R_{mn}} \ldots \).

Lemma 3: If \( \{a_{jk}\} \) satisfies condition (2.1), then, for all \( m, n \geq 0 \),

\[
\sum_{R_{mn}} b_{jk}a_{jk} = \sum_{R_{mn}} B_{jk}A_{11}a_{jk} - \sum_{j=0}^{m} B_{jn}A_{10}a_{j,n+1}
\]

\[
- \sum_{k=0}^{n} B_{mk}A_{01}a_{m+1,k} + B_{mn}a_{m+1,n+1}.
\]
4. Proofs of Theorems 1 and 2. We recall that $D_m(x) = \sum_{j=0}^{m} w_j(x)$ is the Dirichlet kernel, while $F_m(x) = (m + 1)^{-1} \sum_{j=0}^{m} D_j(x)$ is the Fejér kernel for the Walsh system. The following estimates are well known (see [2]):

$$|D_m(x)| < \frac{2}{x} \quad (m = 0, 1, \ldots; 0 < x < 1),$$

and for all integers $m \geq 0$, $p \geq 1$ and for all $0 < x < 1$, except possibly when $x$ is a dyadic rational,

$$(m + 1) |F_m(x)| < \frac{4}{x(x - 2^{-p})} + \frac{4}{x^2} = C(x) \quad \text{if} \quad 2^{-p} < x < 2^{-p+1}.\quad (4.2)$$

Proof of Theorem 1: Pointwise convergence. Let $0 \leq m \leq M$ and $0 \leq n \leq N$. Keeping notation (1.2) in mind, by Lemma 2 we can write that

$$s(Q; x, y) = \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x) D_k(y) A_{11} a_{jk}$$

$$+ \sum_{j=m}^{M} D_j(x) D_N(y) A_{10} a_{j,N+1} - \sum_{j=m}^{M} D_j(x) D_{n-1}(y) A_{10} a_{jn}$$

$$+ \sum_{k=n}^{N} D_M(x) D_k(y) A_{01} a_{M+1,k} - \sum_{k=n}^{N} D_{m-1}(x) D_k(y) A_{01} a_{mk}$$

$$+ a_{M+1,N+1} D_M(x) D_N(y) - a_{M+1,n} D_M(x) D_{n-1}(y)$$

$$- a_{m,N+1} D_{m-1}(x) D_N(y) + a_{mn} D_{m-1}(x) D_{n-1}(y).\quad (4.3)$$

By (4.1), for $0 < x, y < 1$ we get that

$$4^{-1}xy |s(Q; x, y)| \leq \sum_{j=m}^{M} \sum_{k=n}^{N} |A_{11} a_{jk}|$$

$$+ \sum_{j=m}^{M} \sum_{k=n}^{N} (|A_{10} a_{j,N+1}| + |A_{10} a_{jn}|) + \sum_{k=n}^{N} (|A_{01} a_{M+1,k}| + |A_{01} a_{mk}|)$$

$$+ |a_{M+1,N+1}| + |a_{M+1,n}| + |a_{m,N+1}| + |a_{mn}|.$$

Making use of Lemma 1 (with $p = q = 1$) and (2.1), we can see that each term on the right-hand side tends to zero as $\max(m, n) \to \infty$. Thus, the sum $f(x, y)$ of series (1.1) exists for all $0 < x, y < 1$.

$L'(P)$-convergence. It is plain that

$$f(x, y) - s_{mn}(x, y) = \sum_{R_{mn}} a_{jk} w_j(x) w_k(y),$$

where $R_{mn}$ is defined by (3.5). By Lemma 3,

$$f(x, y) - s_{mn}(x, y)$$

$$= \sum_{R_{mn}} D_j(x) D_k(y) A_{11} a_{jk} - \sum_{j=0}^{m} D_j(x) D_n(y) A_{10} a_{j,n+1}$$

$$- \sum_{k=n}^{N} D_m(x) D_k(y) A_{01} a_{m+1,k} - D_m(x) D_n(y) a_{m+1,n+1}.\quad (4.4)$$
Using (4.1) gives, for all \(0 < x, y < 1\),

\[
4^{-1}2xy \left| f(x, y) - s_{mn}(x, y) \right|
\]

\[
\leq \sum_{R=\infty} |A_{11}a_{jk}| + \sum_{j=0}^{m} |A_{10}a_{j,n+1}| + \sum_{k=0}^{n} |A_{01}a_{m+1,k}| + |a_{m+1,n+1}|
\]

\[
\leq 2 \sum_{R=\infty} |A_{11}a_{jk}|
\]

Hence

\[
\left\| f - s_{mn} \right\|_r \leq 8' \left( \sum_{R=\infty} |A_{11}a_{jk}| \right)^r \int_0^1 \int_0^1 \frac{dx}{x^r} \int_0^1 \frac{dy}{y^r}.
\]

Due to (2.2) and \(0 < r < 1\), (2.3) follows immediately.

Proof of Theorem 2: Pointwise convergence. We start with (4.3). We apply Lemma 2 again to the double sum on the right-hand side of (4.3) to obtain

\[
\sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x) D_k(y) \Delta_{11}a_{jk}
\]

\[
= - \sum_{j=m}^{M} \sum_{k=n}^{N} F^*_{j,k}(x, y) \Delta_{21}a_{jk} + \sum_{j=m}^{M} F^*_j(x, y) \Delta_{21}a_{j,N+1}
\]

\[
- \sum_{j=m}^{M} F^*_{j,n-1}(x, y) \Delta_{21}a_{jn} + \sum_{k=n}^{N} F^*_k(x, y) \Delta_{12}a_{M+1,k}
\]

\[
- \sum_{k=n}^{N} F^*_k(x, y) \Delta_{12}a_{mk} + \sum_{j=m}^{M} F^*_j(x, y) \Delta_{11}a_{M+1,N+1}
\]

\[
- \sum_{j=m}^{M} F^*_{j,n-1}(x, y) \Delta_{11}a_{M+1,n} - F^*_{m-1,n-1}(x, y) \Delta_{11}a_{m,n+1}
\]

where

\[
F^*_{mn}(x, y) = (m + 1) (n + 1) F_m(x) F_n(y).
\]

By (4.2), we can conclude for all \(0 < x, y < 1\), except possibly when \(x\) or \(y\) is a dyadic rational,

\[
\left( C(x) C(y) \right)^{-1} \left| \sum_{j=m}^{M} \sum_{k=n}^{N} D_j(x) D_k(y) \Delta_{11}a_{jk} \right|
\]

\[
\leq \sum_{j=m}^{M} \sum_{k=n}^{N} |A_{22}a_{jk}| + \sum_{j=m}^{M} (|A_{21}a_{j,n+1}| + |A_{21}a_{jn}|)
\]

\[
+ \sum_{k=n}^{N} (|A_{12}a_{M+1,k}| + |A_{12}a_{mk}|) + |A_{11}a_{M+1,N+1}|
\]

\[
+ |A_{11}a_{M+1,n}| + |A_{11}a_{m,N+1}| + |A_{11}a_{mn}|
\]

By virtue of Lemma 1 (with \(p = q = 2\)) and (2.1), each term on the right-hand side tends to zero as \(\max (m, n) \to \infty\).

We have four single sums on the right-hand side of (4.3). We claim that each of them tends to zero as \(\max (m, n) \to \infty\), for all \(0 < x, y < 1\). We show this in the
case of the first single sum. A single summation by parts yields

\[ \sum_{j=m}^{M} D_{j}(x) D_{N}(y) A_{10} a_{j,n+1} \]

\[ = \sum_{j=m}^{M} (j + 1) F_{j}(x) D_{N}(y) A_{20} a_{j,n+1} \]

\[ + (M + 1) F_{M}(x) D_{N}(y) A_{10} a_{M,n+1} - m F_{m-1}(x) D_{N}(y) A_{10} a_{m,n+1}. \quad (4.7) \]

Hence, by (4.1) and (4.2), for all \(0 < x, y < 1\), except possibly when \(x\) is a dyadic rational,

\[ \left| \sum_{j=m}^{M} D_{j}(x) D_{N}(y) A_{10} a_{j,n+1} \right| \]

\[ \leq 2y^{-1} C(x) \left\{ \sum_{j=m}^{M} |A_{20} a_{j,n+1}| + |A_{10} a_{M,n+1}| + |A_{10} a_{m,n+1}| \right\}. \]

Thanks to conditions (2.1) and (2.5), each term on the right-hand side tends to zero as \(\max(m, n) \to \infty\). The other three single sums on the right-hand side of (4.3) can be estimated analogously. Finally, by (2.1) and (4.1), the four single terms on the right-hand side of (4.3) tend to zero as \(\max(m, n) \to \infty\), for all \(0 < x, y < 1\).

\(L(P)\)-convergence. Now we start with (4.4). We apply Lemma 3 once more to the double sum on the right-hand side of (4.4). As a result we get that

\[ \sum_{k=0}^{m} D_{j}(x) D_{k}(y) A_{11} a_{j,k} = \sum_{k=0}^{m} F_{jk}(x, y) A_{22} a_{j,k} - \sum_{j=0}^{m} F_{jm}(x, y) A_{21} a_{j,n+1} \]

\[ - \sum_{k=0}^{n} F_{mk}(x, y) A_{12} a_{m+1,k} - F_{mn}(x, y) A_{11} a_{m+1,n+1} \]

where we used notation (4.6). By (4.2),

\[ (C(x) C(y))^{-1} \left| \sum_{k=0}^{m} D_{j}(x) D_{k}(y) A_{11} a_{j,k} \right| \]

\[ \leq \sum_{k=0}^{m} |A_{22} a_{j,k}| + \sum_{j=0}^{m} |A_{21} a_{j,n+1}| + \sum_{k=0}^{n} |A_{12} a_{m+1,k}| + |A_{11} a_{m+1,n+1}| \]

\[ \leq 4 \sum_{k=0}^{m} |A_{22} a_{j,k}|. \quad (4.8) \]

Similarly to (4.7), a single summation by parts gives

\[ \sum_{j=0}^{m} D_{j}(x) D_{n}(y) A_{10} a_{j,n+1} \]

\[ = \sum_{j=0}^{m} (j + 1) F_{j}(x) D_{n}(y) A_{20} a_{j,n+1} + (m + 1) F_{m}(x) D_{n}(y) A_{10} a_{m,n+1} \]

Hence, by (4.1) and (4.2),

\[ \left| \sum_{j=0}^{m} D_{j}(x) D_{n}(y) A_{10} a_{j,n+1} \right| \leq 2y^{-1} C(x) \left\{ \sum_{j=0}^{m} |A_{20} a_{j,n+1}| + |A_{10} a_{m,n+1}| \right\} \]

\[ \leq 2y^{-1} C(x) \sum_{j=0}^{\infty} |A_{20} a_{j,n+1}|. \quad (4.9) \]
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Analogously,
\[ \left| \sum_{k=0}^{\infty} D_m(x) D_k(y) A_{01} a_{m+1,k} \right| \leq 2x^{-1}C(y) \sum_{k=0}^{\infty} |A_{02} a_{m+1,k}|. \quad (4.10) \]

Combining (4.4), (4.8)–(4.10) and (2.4)–(2.6) yields (2.3) for all \( 0 < r < \frac{1}{2} \) if we take into account that, by (4.2),
\[
\int_0^1 C(x) \, dx \leq \sum_{p=1}^{\infty} \int_0^{2^{-p-1}} \left( 4^{p-x} (x - 2^{-p}) \right) \, dx + \int_0^{2^{-p}} \left( 4^{p-x} \right) \, dx
\leq \sum_{p=1}^{\infty} \left( \frac{4^p}{1 - r} \right) 2^{p(1 - r)} + 4^p (1 - 2r) < \infty.
\]

5. Concluding remarks. In the case of Theorem 1(ii) we can prove somewhat more than (2.3) for \( 0 < r < 1 \). To present this, let "meas" denote the planar Lebesgue measure and let \( \ln u = \max (1, \ln u) \).

**Theorem 3:** If a double sequence \( \mathcal{A} = \{a_{jk}\} \) satisfies conditions (2.1) and (2.2), then, for every \( \varepsilon > 0 \),
\[
\mu = \text{meas} \left\{ (x, y) \in I^2 : \sup_{m,n \geq 0} |\sigma_{mn}(x, y)| \geq \varepsilon \right\} \leq \frac{4||\mathcal{A}||}{\varepsilon} \left( 1 + \ln \frac{\ln u}{4||\mathcal{A}||} \right), \quad (5.1)
\]
where
\[
||\mathcal{A}\| = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |A_{11} a_{jk}|. \quad (5.2)
\]

Since the space \( c_0 \cap BV_2 \) of double null sequences of bounded variation endowed with norm (5.2) is a Banach space, condition (5.1) is only slightly weaker than the condition that the mapping \( \mathcal{A} \rightarrow f \) is of weak type \( (1, 1) \), where \( f = f(x, y) \) is the sum of series (1.1) (see Theorem 1).

**Proof of Theorem 3:** Similarly to (4.3),
\[
\sigma_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} D_j(x) D_k(y) A_{11} a_{jk} + \sum_{j=0}^{m} D_j(x) D_n(y) A_{10} a_{j,n+1}
+ \sum_{k=0}^{n} D_m(x) D_k(y) A_{01} a_{m+1,k} + a_{m+1,n+1} D_m(x) D_n(y).
\]

By (4.1), for all \( 0 < x, y < 1 \), we get that
\[
4^{-1} xy \ |\sigma_{mn}(x, y)| \leq \sum_{j=0}^{m} \sum_{k=0}^{n} |A_{11} a_{jk}| + \sum_{j=0}^{m} |A_{10} a_{j,n+1}|
+ \sum_{k=0}^{n} |A_{01} a_{m+1,k}| + |a_{m+1,n+1}|,
\]
whence, by (2.1) (cf. the proof of Lemma 1),
\[
4^{-1} xy \ |\sigma_{mn}(x, y)| \leq \sum_{j=0}^{m} \sum_{k=0}^{n} |A_{11} a_{jk}| + \sum_{j=0}^{m} \sum_{k=n+1}^{\infty} |A_{11} a_{jk}|
+ \sum_{j=m+1}^{\infty} \sum_{k=0}^{n} |A_{11} a_{jk}| + \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} |A_{11} a_{jk}| = ||\mathcal{A}||.
\]

Now for every \( \varepsilon \geq 4 ||\mathcal{A}||, \mu \leq \text{meas} \left\{ (x, y) \in I^2 : xy \leq \gamma \right\} = \gamma + \gamma \ln (1/\gamma) \), where \( \gamma = 4 ||\mathcal{A}||/\varepsilon \).
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