Bouligand–Severi tangents in MV-algebras

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Abstract. In their important recent paper published in the Annals of Pure and Applied Logic, Dubuc and Poveda call an MV-algebra $A$ strongly semisimple if all principal quotients of $A$ are semisimple. All boolean algebras are strongly semisimple, and so are all finitely presented MV-algebras. We show that for any 1-generator MV-algebra, semisimplicity is equivalent to strong semisimplicity. Further, a semisimple 2-generator MV-algebra $A$ is strongly semisimple if and only if its maximal spectral space $\mu(A) \subseteq [0,1]^2$ does not have any rational Bouligand–Severi tangents at its rational points. In general, when $A$ is finitely generated and $\mu(A) \subseteq [0,1]^n$ has a Bouligand–Severi tangent then $A$ is not strongly semisimple. An MV-algebra $A$ is strongly semisimple if and only if so is every 2-generator subalgebra of $A$.

1. Introduction

We refer to [4] and [8] for background on MV-algebras. Following Dubuc and Poveda [5], we say that an MV-algebra $A$ is strongly semisimple if for every principal ideal $I$ of $A$ the quotient $A/I$ is semisimple. Since $\{0\}$ is a principal ideal of $A$, every strongly semisimple MV-algebra is semisimple. The definition of “logically complete” MV-algebras in [1] is a variant of this notion, where one further assumes $I \neq \{0\}$. The paper [7] is devoted to the frame-theoretic variant of strongly semisimple MV-algebras, called “Yosida frames”. These papers, together with the results of the present paper, show that strong semisimplicity is a very interesting purely algebraic counterpart of the simplicial, topological, and differential structure of MV-algebras. Further, from the logical viewpoint, 4.3 in [9] shows that strongly semisimple MV-algebras coincide with Lindenbaum algebras of theories $\Theta$ in infinite-valued Lukasiewicz logic having the following property: for any formula $\psi$, the set of syntactic consequences of $\Theta \cup \{\psi\}$ coincides with the set of (Bolzano–Tarski) semantic consequences of $\Theta \cup \{\psi\}$. 

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From a classical result of Hay [6] and Wójtikí [14] (see also 4.6.7 in [4] and 1.6 in [8]), it follows that every finitely presented MV-algebra is strongly semisimple. Trivially, all hyperarchimedean MV-algebras, hence in particular all boolean algebras, are strongly semisimple, and so are all simple and all finite MV-algebras (see 3.5 and 3.6.5 in [4]).

For any real-valued function $g$ we will write $Zg = g^{-1}(0)$ for its zero set.

Our paper is devoted to $n$-generator strongly semisimple MV-algebras. When $n = 1$, strong semisimplicity is equivalent to semisimplicity (Theorem 5.1). To deal with the general case, we first recall that the free $n$-generator MV-algebra is the MV-algebra $M([0,1]^n)$ of all McNaughton functions $f : [0,1]^n \to [0,1]$, with pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. See 9.1.5 in [4].

For any nonempty closed set $X \subseteq [0,1]^n$ we let $M(X)$ denote the MV-algebra of restrictions to $X$ of the functions in $M([0,1]^n)$.

By 3.6.7 in [4], $M(X)$ is a semisimple MV-algebra; actually, up to isomorphism, $M(X)$ is the most general possible $n$-generator semisimple MV-algebra $A$. To see this, pick generators $\{a_1, \ldots, a_n\}$ of $A$. Let $\pi_i : [0,1]^n \to [0,1]$ be the projection functions in the free MV-algebra $M([0,1]^n)$ for $i = 1, \ldots, n$. Then the assignment that maps $\pi_i \mapsto a_i$ for each $i = 1, \ldots, n$, uniquely extends to a homomorphism $\eta_a : M([0,1]^n) \to A$ of the free $n$-generator MV-algebra onto $A$. Let $h_a = \ker(\eta_a)$ be the kernel of this homomorphism and let

\begin{equation}
Z_a = \bigcap \{Zf \mid f \in h_a\}
\end{equation}

be the intersection of the zero sets of the McNaughton functions in $h_a$. Then

\begin{equation}
A \cong M(Z_a).
\end{equation}

A point $x \in \mathbb{R}^n$ is said to be rational if so are all its coordinates. By a rational vector we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w \subseteq \mathbb{R}^n$ contains at least two rational points. An MV-algebra $A$ is strongly semisimple if and only if so is every 2-generator subalgebra of $A$ (Proposition 4.1). A 2-generator MV-algebra $A = M(X)$, with nonempty closed $X \subseteq [0,1]^2$, is strongly semisimple if and only if $X$ has no rational outgoing Bouligand–Severi tangent vector at any of its rational points, [2], [12], and [10]. See Theorem 3.1. As proved in Theorem 2.3, for any closed $X \subseteq [0,1]^n$, having such a tangent is a condition sufficient for $M(X)$ not to be strongly semisimple.

**Notation.** Following p. 33 in [4] or p. 21 in [8], for $k \in \mathbb{N}$, $k \cdot g$ stands for the $k$-fold pointwise truncated addition of $g$.

**2. Strong semisimplicity and Bouligand–Severi tangents**

Severi (see §53, p. 59 and p. 392 of [11], as well as §1, p. 99 of [12]) and independently, Bouligand (p. 32 in [2]) called a half-line $H \subseteq \mathbb{R}^n$ tangent to a set $X \subseteq \mathbb{R}^n$ at an accumulation point $x$ of $X$ if for all $\epsilon, \delta > 0$ there is $y \in X$ different from $x$
such that $||y - x|| < \epsilon$, and the angle between $H$ and the half-line through $y$ originating at $x$ is $\delta < \delta$. Here as usual, $||v||$ is the length of the vector $v \in \mathbb{R}^n$.

On §2, p. 100 and §4, p. 102 of [12], Severi noted that for any accumulation point $x$ of a closed set $X$ there is a half-line $H$ tangent to $X$ at $x$.

Today (see, e.g., p. 16 in [3], or p. 1376 in [10]), Bouligand–Severi tangents are routinely defined as follows.

**Definition 2.1.** Let $x$ be an element of a closed subset $X$ of $\mathbb{R}^n$, and $u$ a unit vector in $\mathbb{R}^n$. We then say that $u$ is a Bouligand–Severi tangent (unit) vector to $X$ at $x$ if $X$ contains a sequence $x_0,x_1,\ldots$ of elements, all different from $x$, such that

$$
\lim_{i \to \infty} x_i = x \quad \text{and} \quad \lim_{i \to \infty} (x_i - x)/||x_i - x|| = u.
$$

Observe that $x$ is an accumulation point of $X$. We further say that $u$ is outgoing if for some $\lambda > 0$ the segment $\text{conv}(x,x + \lambda u)$ intersects $X$ only at $x$.

Already Severi noted that his definition of tangent half-line $H = x + \mathbb{R}_{\geq 0}u$ is equivalent to Definition 2.1. More precisely:

**Proposition 2.2.** (§5, p. 103 of [12]). For any nonempty closed subset $X$ of $\mathbb{R}^n$, point $x \in X$, and unit vector $u \in \mathbb{R}^n$ the following conditions are equivalent:

(i) For all $\epsilon, \delta > 0$, the cone $\text{cone}_{x,u,\epsilon,\delta}$ with apex $x$, axis parallel to $u$, vertex angle $2\delta$ and height $\epsilon$ contains infinitely many points of $X$.

(ii) $u$ is a Bouligand–Severi tangent vector to $X$ at $x$.

When $n = 1$, $\text{cone}_{x,u,\epsilon,\delta}$ is the segment $\text{conv}(x,x + \epsilon u)$. When $n = 2$, $\text{cone}_{x,u,\epsilon,\delta}$ is the isosceles triangle $\text{conv}(x,a,b)$ with vertex $x$, basis $\text{conv}(a,b)$, height equal to $\epsilon$ (and parallel to $u$), and vertex angle $\alpha \hat{a} \hat{b} = 2\delta$.

The next two results provide necessary and sufficient geometric conditions on $X$ for the semisimple MV-algebra $M(X)$ to be strongly semisimple. These conditions are stated in terms of the nonexistence of Bouligand–Severi tangent vectors having certain rationality properties.

**Theorem 2.3.** Let $X$ be a nonempty closed set in $[0,1]^n$. Suppose $X$ has a Bouligand–Severi rational outgoing tangent vector $u$ at some rational point $x \in X$. Then $M(X)$ is not strongly semisimple.

**Proof.** Since $u$ is outgoing, let $\lambda > 0$ satisfy $X \cap \text{conv}(x,x + \lambda u) = \{x\}$. Without loss of generality $x + \lambda u \in \mathbb{Q}^n$. By Definition 2.1, our hypothesis yields a sequence $w_1, w_2, \ldots$ of distinct points of $X$, all distinct from $x$, accumulating at $x$, at strictly decreasing distances from $x$, in such a way that the sequence of unit vectors $u_i$ given by $(w_i - x)/||w_i - x||$ tends to $u$ as $i$ tends to $\infty$. Let $y = x + \lambda u$. Since $X \cap \text{conv}(x,y) = \{x\}$, no point $w_i$ lies on the segment $\text{conv}(x,y)$, and we can further assume that the sequence of angles $\hat{w}_i \hat{x} \hat{y}$ is strictly decreasing and tends to zero as $i$ tends to $\infty$.

Since both points $x$ and $y$ are rational, by 2.10 in [8], for some $g \in M([0,1]^n)$ the zero set

$$Zg = \{z \in [0,1]^n \mid g(z) = 0\}$$
the value of the incremental ratio \( \frac{\partial g(x)}{\partial (u)} \). Thus,
\[
\frac{\partial g(x)}{\partial (u)} = 0.
\]

Let \( J \) be the ideal of \( \mathcal{M}([0,1]^n) \) generated by \( g \),
\[
J = \{ f \in \mathcal{M}([0,1]^n) \mid f \leq k \cdot g \text{ for some } k = 0,1,2,\ldots \}.
\]

Then for each \( f \in J \),
\[
\frac{\partial f(x)}{\partial (u)} = 0.
\]

Since the directional derivatives of \( f \) at \( x \) are continuous (meaning that the map \( t \mapsto \frac{\partial f(x)}{\partial t} \) is continuous), it follows that
\[
\lim_{t \to u} \frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial u} = 0. \tag{2.1}
\]

Let \( g' = g \upharpoonright X \) and let
\[
J' = \{ f' \in \mathcal{M}(X) \mid f' \leq k \cdot g' \text{ for some } k = 0,1,2,\ldots \}
\]
be the ideal of \( \mathcal{M}(X) \) generated by \( g' \). A moment’s reflection shows that
\[
J' = \{ l \upharpoonright X \mid l \in J \}. \tag{2.2}
\]

One inclusion is trivial. For the converse inclusion, if \( f \upharpoonright X \leq (k \cdot g) \upharpoonright X \) then letting \( l = f \wedge k \cdot g \) we get \( l \leq k \cdot g \). So \( l \in J \) and \( l \upharpoonright X = f \upharpoonright X \), whence \( f \upharpoonright X \) is extendible to some \( l \in J \).

For any \( f \in \mathcal{M}([0,1]^n) \), the piecewise linearity of \( f \) ensures that for all large \( i \) the value of the incremental ratio \( (f(w_i) - f(x))/||w_i - x|| \) coincides with the directional derivative \( \frac{\partial f(x)}{\partial u} \) along the unit vector \( u_i = (w_i - x)/||w_i - x|| \). Thus in particular, if \( f \upharpoonright X = f' \in J' \), from (2.1)–(2.2) it follows that
\[
\lim_{i \to \infty} \frac{f(w_i) - f(x)}{||w_i - x||} = 0.
\]

Since \( x \) is rational, again by 2.10 in [8] there is \( j \in \mathcal{M}([0,1]^n) \) with \( Z_j = \{ x \} \). For some \( \omega > 0 \) we have \( \frac{\partial j(x)}{\partial (u)} = \omega \), whence
\[
\lim_{i \to \infty} \frac{j(w_i) - j(x)}{||w_i - x||} = \omega.
\]

Therefore, \( j' \notin J' \). Since \( Z_j \cap X = \{ x \} \), recalling 4.19 in [8] we see that the only maximal ideal of \( \mathcal{M}(X) \) containing \( J \) is the set of all functions in \( \mathcal{M}(X) \) that vanish at \( x \). Thus, \( j' \) belongs to all maximal ideals of \( \mathcal{M}(X) \) containing \( J \).

By 3.6.6 in [4], \( \mathcal{M}(X) \) is not strongly semisimple; specifically, \( j' / J' \) is infinitesimal in the principal quotient \( \mathcal{M}(X) / J' \).
3. A partial converse of Theorem 2.3

**Theorem 3.1.** Let \( X \subseteq [0, 1]^n \) be a nonempty closed set. Suppose the MV-algebra \( \mathcal{M}(X) \) is not strongly semisimple.

(i) Then \( X \) has a Bouligand–Severi tangent vector \( u \) at some point \( x \in X \) satisfying the following nonalignment condition: there is a sequence of distinct \( w_i \in X \), all distinct from \( x \) such that

\[
\lim_{i \to \infty} w_i = x, \quad \lim_{i \to \infty} \frac{w_i - x}{||w_i - x||} = u, \quad w_i \notin \text{conv}(x, x + u) \text{ for all } i.
\]

(ii) In particular, if \( n = 2 \), then \( X \) has a Bouligand–Severi outgoing rational tangent vector \( u \) at some rational point \( x \in X \).

**Proof.** (i) The hypothesis yields a function \( g \in \mathcal{M}([0, 1]^n) \), with its restriction \( g' = g|_X \in \mathcal{M}(X) \), in such a way that the principal ideal \( J' \) of \( \mathcal{M}(X) \) generated by \( g' \),

\[
J' = \{ l' \in \mathcal{M}(X) \mid l' \leq k \cdot g' \text{ for some } k = 1, 2, \ldots \}
\]

is strictly contained in the intersection \( I \) of all maximal ideals of \( \mathcal{M}(X) \) containing \( J' \). Thus for some \( j \in \mathcal{M}([0, 1]^n) \) letting \( j' = j|_X \) we have \( j' \in I \cap J' \). By 3.6.6 in [4] and 4.19 in [8],

\[
j' = 0 \text{ on } Zg', \text{ i.e., } X \cap Zj \supseteq X \cap Zg
\]

and

\[
\forall m = 0, 1, \ldots, \exists z_m \in X, \ j'(z_m) > m \cdot g'(z_m).
\]

There is a sequence of integers \( 0 < m_0 < m_1 < \ldots \) and a subsequence \( y_0, y_1, \ldots \) of \( \{ z_1, z_2, \ldots \} \) such that \( y_i \neq y_l \) for \( i \neq l \) and

\[
\forall t = 0, 1, \ldots, \ j'(y_t) > m_t \cdot g'(y_t).
\]

The compactness of \( X \) yields an accumulation point \( x \in X \) of the \( y_t \). Without loss of generality (taking a subsequence, if necessary) we can further assume

\[
||y_0 - x|| > ||y_1 - x|| > \cdots, \text{ whence } \lim_{i \to \infty} y_i = x.
\]

By (3.3), for all \( t \), \( j'(y_t) > 0 \). Then by (3.1), \( g'(y_t) > 0 \). For each \( i = 0, 1, \ldots, \) defining the unit vector \( u_i \in \mathbb{R}^n \) by \( u_i = (y_i - x)/||y_i - x|| \), we obtain a sequence of (possibly repeated) unit vectors \( u_i \in \mathbb{R}^n \). Since the boundary of the unit ball in \( \mathbb{R}^n \) is compact, some unit vector \( u \in \mathbb{R}^n \) satisfies

\[
\forall \epsilon > 0 \text{ there are infinitely many } i \text{ such that } ||u_i - u|| < \epsilon.
\]

Some subsequence \( w_0, w_1, \ldots \) of the \( y_t \) will satisfy the condition

\[
\forall \epsilon, \delta > 0 \text{ there is } k \text{ such that for all } i > k, \ w_i \in \text{conv}_x, u, \epsilon, \delta.
\]
Correspondingly, the sequence \( v_0, v_1, \ldots \) given by \( v_k = (w_k - x)/\|w_k - x\| \) will satisfy

\[
\lim_{i \to \infty} v_i = u.
\]

We have just proved that \( u \) is a Bouligand–Severi tangent to \( X \) at \( x \).

To complete the proof of (i) we need the following:

**Fact 1.** \( g'(x) = 0 \).

Otherwise, from the continuity of \( g \), for some real \( \rho > 0 \) and suitably small \( \epsilon > 0 \), we have the inequality \( g(z) > \rho \) for all \( z \) in the open ball \( B_{x,\epsilon} \) of radius \( \epsilon \) centered at \( x \). By (3.5), \( B_{x,\epsilon} \) contains infinitely many \( w_i \). There is a fixed integer \( \bar{m} > 0 \) such that \( 1 = \bar{m} \cdot g \geq j \) for all these \( w_i \), which contradicts (3.3).

**Fact 2.** \( j'(x) = 0 \).

This immediately follows from (3.1) and Fact 1.

**Fact 3.** \( \partial g(x)/\partial u = 0 \).

Aiming at a contradiction, suppose \( \partial g(x)/\partial u = \theta > 0 \). In view of the continuity of the map \( t \mapsto \partial g(x)/\partial t \), let \( \delta > 0 \) be such that \( \partial g(x)/\partial r > \theta/2 \), for any unit vector \( r \) such that \( \bar{r}u < \delta \). Since, by Fact 2, \( j(x) = 0 \) and both \( g \) and \( j \) are piecewise linear, there is an \( \epsilon > 0 \) together with an integer \( \bar{k} > 0 \) such that \( \bar{k} \cdot g \geq j \) over the cone \( C = \text{cone}_{x,u,\epsilon,\delta} \). By (3.5), \( C \) contains infinitely many \( w_i \), in contradiction with (3.3).

To conclude the proof of the nonalignment condition in (i), it is sufficient to show the following:

**Fact 4.** There is \( \lambda > 0 \) such that for all large \( i \) the segment \( \text{conv}(x, x + \lambda u) \) contains no \( w_i \).

For otherwise, from Fact 3, \( \partial g(x)/\partial (u) = 0 \), whence the piecewise linearity of \( g \) ensures that \( g \) vanishes on infinitely many \( w_i \) of \( \text{conv}(x, x + \lambda u) \) arbitrarily near \( x \). Any such \( w_i \) belongs to \( X \). Hence, by (3.1), \( j(w_i) = 0 \), in contradiction with (3.3).

The proof of (i) is now complete.

(ii) Let \( H^\pm \) be the two closed half-spaces of \( \mathbb{R}^2 \) determined by the line passing through \( x \) and \( x + u \). By (3.5), infinitely many \( w_i \) lie in the same closed half-space, say, \( H^+ \). Without loss of generality, \( H^+ \cap \text{int}([0,1]^2) \neq \emptyset \). Let \( u^\perp \) be the vector orthogonal to \( u \) such that \( x + u^\perp \in H^+ \).

**Fact 5.** For all small \( \epsilon > 0 \),

\[
\frac{\partial g(x + \epsilon u)}{\partial u^\perp} > 0.
\]
Aiming at a contradiction, assume $\partial g(x + \epsilon u)/\partial u^\perp = 0$. Since $g$ is piecewise linear, by Facts 1 and 3, for suitably small $\eta, \omega > 0$, the function $g$ vanishes over the triangle $T = \text{conv}(x, x + \eta u, x + \eta u + \omega u^\perp)$. By (3.5), $T$ contains infinitely many $w_i$. By (3.1), $g(w_i) = j(w_i) = 0$, contradicting (3.3).

**Fact 6.**

$$\frac{\partial j(x)}{\partial u} > 0.$$

Otherwise, $\partial j(x)/\partial u = 0$. Fact 5 yields a fixed integer $\bar{h}$ such that, on a suitably small triangle of the form $T = \text{conv}(x, x + \epsilon u, x + \epsilon u + \omega u^\perp)$, we have $\bar{h} \cdot g \geq j$. By (3.5), $T$ contains infinitely many $w_i$, again contradicting (3.3).

We now prove a strong form of Fact 4, showing that $u$ is an outgoing tangent vector:

**Fact 7.** For some $\lambda > 0$ the segment $\text{conv}(x, x + \lambda u)$ intersects $X$ only at $x$.

Otherwise, from Facts 1 and 3 it follows that $g$ vanishes on infinitely many points of $X \cap \text{conv}(x, x + \lambda u)$ converging to $x$. By (3.1), $j'$ vanishes on all these points. Since $j$ is piecewise linear, $\partial j(x)/\partial u = 0$, contradicting Fact 6.

By a rational line in $\mathbb{R}^n$ we mean a line passing through at least two distinct rational points.

**Fact 8.** $x$ is a rational point, and $u$ is a rational vector.

As a matter of fact, Facts 6 and 2 yield a rational line $L$ through $x$. On the other hand, Facts 3 and 5 show that the line passing through $x$ and $x + u$ is rational and different from $L$. Thus $x$ is rational, hence so is the vector $u$.

We conclude that $X$ has $u$ as a Bouligand–Severi outgoing rational tangent vector at the rational point $x$.

Figure 1 is a sketch of the functions $g$ and $j$ in the foregoing proof.

Recalling Theorem 2.3 we now obtain:

**Corollary 3.2.** Let $X \subseteq [0, 1]^2$ be a nonempty closed set. Then $\mathcal{M}(X)$ is not strongly semisimple iff $X$ has a Bouligand–Severi outgoing rational tangent vector $u$ at some rational point $x \in X$.

**Examples.** The above corollary provides many examples of 2-generator strongly semisimple MV-algebras:

(i) Let $\kappa \in [0, 1]$ be irrational. Let $W$ be the arc of parabola $\{(x, y) \in [0, 1]^2 | y = \kappa x^2\}$. Then $\mathcal{M}(W)$ is strongly semisimple – for want of rational points in $W$.

One can similarly construct 2-generator strongly semisimple MV-algebras of the form $\mathcal{M}(V)$, by letting $V$ be a closed subset of $[0, 1]^2$ without rational points, or else, without outgoing rational tangents.

(ii) Following [13], let $Q \subseteq [0, 1]^2$ be a polyhedron in $[0, 1]^2$, i.e., a finite union of $m$-simplexes ($m = 0, 1, 2$) in $[0, 1]^2$. Then $Q$ does not have any outgoing Bouligand–Severi tangent, whence $\mathcal{M}(Q)$ is strongly semisimple.
Figure 1. A Bouligand–Severi outgoing tangent vector $u$ to $X$ at $x$, and two functions $g$ and $j$. The restriction $g \mid X$ generates a principal ideal $J'$ of $\mathcal{M}(X)$. The restriction $j \mid X$ does not belong to $J'$, but belongs to the only maximal ideal $I'$ of $\mathcal{M}(X)$ containing $J'$, namely the set of all functions in $\mathcal{M}(X)$ vanishing at $x$. So the principal quotient $\mathcal{M}(X)/J'$ is not semisimple.

(iii) (Generalizing (ii)). Let $A$ be a 2-generator subalgebra of a semisimple tensor product (see §9.4 in [8]) of the form $[0,1] \otimes D$, where $D$ is a finitely presented MV-algebra. Using Lemma 3.6 and Theorem 6.3 in [8], one sees that $A$ is isomorphic to an MV-algebra of the form $\mathcal{M}(Q)$ for some polyhedron $Q \subseteq [0,1]^2$. Thus $A$ is strongly semisimple.

4. The general case

The central role of finitely generated, and especially of 2-generator strongly semisimple MV-algebras among all strongly semisimple MV-algebras, is shown by the following result:

**Proposition 4.1.** For any MV-algebra $A$ the following conditions are equivalent:

(i) $A$ is strongly semisimple;

(ii) $A$ is the direct limit of a direct system $\mathcal{S} = \{A_i, \phi_{ij}\}$ of finitely generated strongly semisimple algebras $A_i$, where all the homomorphisms $\phi_{ij} : A_i \rightarrow A_j$ are embeddings;

(iii) each 2-generator subalgebra of $A$ is strongly semisimple.

**Proof.** Recall that an MV-algebra is semisimple if and only if it has no infinitesimals. For any MV-algebras $C$ and $D$, and embedding $\phi : C \rightarrow D$, letting, for any $y \in C$, $(y)_D$ denote the ideal generated by $y$ in $C$, we first make the following elementary observations:
(I) For each $c \in C$, the map $\phi: C/\langle c \rangle_C \to D/\langle \phi(c) \rangle_D$ defined by $x/\langle c \rangle_C \mapsto \phi(x)/\langle \phi(c) \rangle_D$ is an embedding. This immediately follows by observing that $\phi(\langle c \rangle_C) = \langle \phi(c) \rangle_D \cap \phi(C)$.

(II) $c \in C$ is an infinitesimal of $C$ if and only if $\phi(c)$ is an infinitesimal of $D$.

(III) If $D$ is strongly semisimple then so is $C$. As a matter of fact, for any $c \in C$, the map $\phi: C/\langle c \rangle_C \to D/\langle \phi(c) \rangle_D$ of (I) is an embedding. By hypothesis, $D/\langle \phi(c) \rangle_D$ is semisimple, whence so is $C/\langle c \rangle_C$ by (II).

We are now ready to prove the proposition.

(i)$\Rightarrow$(ii). Let $\mathcal{A} = \{A_i \subseteq A \mid A_i$ is a finitely generated subalgebra of $A\}$, and let $\phi_{ij}: A_i \to A_j$ be the inclusion map whenever $A_i \subseteq A_j$. Then $A$ together with the homomorphisms $\phi_{ij}$ is a direct system of MV-algebras, having $A$ as its direct limit. By (III), each $A_i$ is strongly semisimple.

(ii)$\Rightarrow$(i). Let $\mathcal{S} = \{A_i, \phi_{ij}\}$ be a directed system of strongly semisimple MV-algebras, indexed by the directed partially ordered set $I$, where each $\phi_{ij}$ is an embedding of $A_i$ into $A_j$. Let $A$ be the direct limit of $\mathcal{S}$ with the telescopic maps $\phi_{i\infty}: A_i \to A$. Each $\phi_{i\infty}$ is an embedding. Suppose that $A$ is not strongly semisimple, (absurdum hypothesis), and let $g \in A$ be such that $A/\langle g \rangle_A$ is not semisimple. Then there is an element $e \in A$ such that $e/\langle g \rangle_A$ is an infinitesimal of $A/\langle g \rangle_A$. Since the partial order of the index set $I$ is directed, for some $i \in I$ there are $g_i, e_i \in A_i$ with $\phi_{i\infty}(g_i) = g$ and $\phi_{i\infty}(e_i) = e$. The map $\phi_{i\infty}: A_i/\langle g_i \rangle_{A_i} \to A/\langle g \rangle_A$ of (I) is an embedding. By (II), $e_i/\langle g_i \rangle_{A_i}$ is an infinitesimal element of $A_i/\langle g_i \rangle_{A_i}$, contrary to the hypothesis that $A_i$ is strongly semisimple.

(i)$\Rightarrow$(iii). Immediate from (III).

(iii)$\Rightarrow$(i). If $A$ is not strongly semisimple there are elements $g, e \in A$ such that $e/\langle g \rangle_A$ is an infinitesimal in $A/\langle g \rangle_A$. Let $B \subseteq A$ be the subalgebra of $A$ generated by $g$ and $e$. By (I) and (II), $e/\langle g \rangle_B$ is an infinitesimal element of $B/\langle g \rangle_B$, and $B$ is not strongly semisimple. \qed

5. Coda: one-generator MV-algebras

The following result is an easy consequence of Theorem 3.1. We include the elementary proof because it provides a technique for dealing with strong semisimplicity independently of Bouligand–Severi tangents.

**Theorem 5.1.** Every one-generator semisimple MV-algebra $A$ is strongly semisimple.

**Proof.** As in (1.1)–(1.2), let $X \subseteq [0, 1]$ be a nonempty closed set such that $A \cong \mathcal{M}(X)$. For some $g \in \mathcal{M}([0, 1])$ let $J$ be the principal ideal of $\mathcal{M}([0, 1])$ generated by $g$, and let $J'$ be the principal ideal of $\mathcal{M}(X)$ generated by $g' = g | X$.

The short argument immediately following (2.2) shows that $J' = \{f | X \mid l \in J\}$. For every $f \in \mathcal{M}([0, 1])$, letting $f' = f | X$ we must prove: if $f'$ belongs to all
maximal ideals of $M(X)$ to which $g'$ belongs, then $f'$ belongs to $J'$. By 3.6.6 in [4] and 4.19 in [8], this amounts to proving

\[(5.1) \quad \text{if } f = 0 \text{ on } Zg \cap X, \text{ then } f \upharpoonright X \in J'.\]

Let $\Delta$ be a triangulation of $[0,1]$ such that $f$ and $g$ are linear over every simplex of $\Delta$. The existence of $\Delta$ follows from the piecewise linearity of $f$ and $g$, [13]. In view of the compactness of $X$ and $[0,1]$, it is sufficient to settle the following:

Claim. Suppose $f \in M([0,1])$ vanishes over $Zg \cap X$. Then for all $x \in X$ there is an open neighbourhood $N_x \ni x$ in $[0,1]$ together with an integer $m_x \geq 0$ such that $m_x \cdot g \geq f$ on $N_x \cap X$.

We proceed by cases.

Case 1. $g(x) > 0$. Then for some integer $r$ and open neighbourhood $N_x \ni x$ we have $g > 1/r$ on $N_x$. Letting $m_x = r$ we have $1 = m_x \cdot g \geq f$ on $N_x$, whence a fortiori, $m_x \cdot g \geq f$ on $N_x \cap X$.

Case 2. $g(x) = 0$. Since $f$ vanishes on $Zg \cap X$, then $f(x) = 0$. Let $T$ be a 1-simplex of $\Delta$ such that $x \in T$. Let $T_x$ be the smallest face of $T$ containing $x$.

Subcase 2.1. $T_x = T$. Then $x \in \text{int}(T)$. Since $g$ is linear over $T$ $g$ vanishes on $T$. By our hypotheses on $f$ and $\Delta$, $f$ vanishes on $T$, whence $0 = g \geq f = 0$ on $T$. Letting $N_x = \text{int}(T)$ and $m_x = 1$, we get $m_x \cdot g \geq f$ on $N_x$ whence a fortiori, the inequality holds on $N_x \cap X$.

Subcase 2.2. $T_x = \{x\}$. Then $T = \text{conv}(x,y)$ for some $y \neq x$. Without loss of generality, $y > x$. We will exhibit a right open neighbourhood $R_x \ni x$ and an integer $r_x \geq 0$ such that $r_x \cdot g \geq f$ on $R_x \cap X$. The same argument yields a left neighbourhood $L_x \ni x$ and an integer $l_x \geq 0$ such that $l_x \cdot g \geq f$ on $L_x \cap X$. One then takes $N_x = R_x \cup L_x$ and $m_x = \max(r_x,l_x)$.

Subsubcase 2.2.1. If both $g$ and $f$ vanish at $y$, then they vanish on $T$ (because they are linear on $T$). Defining $R_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$, we get $r_x \cdot g \geq f$ on $R_x$, whence, in particular, on $R_x \cap X$.

Subsubcase 2.2.2. If both $g$ and $f$ are positive at $y$, then for all suitably large $m$ we have $m \cdot g \geq f$ on $T$ because $f(x) = 0$ and both $f$ and $g$ are linear on $T$. Letting $r_x$ be the smallest such $m$ and letting $R_x = \text{int}(T) \cup \{x\}$, we have the desired inequality on $R_x$, and a fortiori on $R_x \cap X$.

Subsubcase 2.2.3. $g(y) = 0, f(y) > 0$. By our hypotheses on $\Delta$, $g$ is linear on $T$ and hence $g = 0$ on $T$. It follows that $X \cap T = \{x\}$; for otherwise, our assumption $Zf \cap X \supseteq Zg \cap X$ together with the linearity of $f$ on $T$ would imply $f(y) = 0$, contrary to our current hypothesis. Letting $R_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$ we have $r_x \cdot g \geq f$ on $R_x \cap X$. \hfill \Box

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