\(\alpha\)-isoptics of a triangle and their connection
to \(\alpha\)-isoptic of an oval

MALGORZATA MICHALSKA (*) - WITOLD MOZGAWA (**)

ABSTRACT - For a fixed positive angle \(\alpha, \alpha < \pi\) we get an explicit formulas for an \(\alpha\)-isoptic curve of a triangle and study some of its properties. We use obtained results to show that \(\alpha\)-isoptic of an oval is an envelope of \(\alpha\)-isoptics of properly chosen triangles.

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1. Introduction

For two nontrivial vectors in the complex plane \(u = u_1 + iu_2, v = v_1 + iv_2\) let \([u, v] = u_1v_2 - u_2v_1\). On the other hand we know that \([u, v] = |u| \cdot |v| \sin \mathcal{L}(u, v)\). Thus, we have the useful formula

\[
\sin \mathcal{L}(u, v) = \frac{[u, v]}{|u| \cdot |v|}.
\]

An \(\alpha\)-isoptic curve \(C_\alpha\) of a plane, closed, convex curve \(C\) is a set of those points in the complex plane from which the curve \(C\) is seen under a fixed angle \(\pi - \alpha\), \(\alpha \in (0, \pi)\). If \(C\) is strictly convex and the origin of the plane is

(*) Indirizzo dell’A.: Institute of Mathematics, Maria Curie-Skłodowska University, pl. M. Curie-Skłodowska 1, 20-031 Lublin, Poland.
E-mail: malgorzata.michalska@poczta.umcs.lublin.pl

(**) Indirizzo dell’A.: Institute of Mathematics, Maria Curie-Skłodowska University, pl. M. Curie-Skłodowska 1, 20-031 Lublin, Poland.
E-mail: mozgawa@poczta.umcs.lublin.pl
chosen inside $C$, then there exists (cf. [2]) a differentiable function $p$ such that $p(t), \ t \in [0, 2\pi]$ is the distance from the origin to the support line. Function $p$ is called a support function and in its terms we have the parametrization of $C$

$$(1.2) \quad z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}, \quad t \in [0, 2\pi],$$

and the parametrization of its $z$-isoptic

$$(1.3) \quad z_z(t) = p(t)e^{it} + \left\{ -p(t) \cot z + \frac{1}{\sin z} p(t + z) \right\} ie^{it}, \quad t \in [0, 2\pi].$$

Properties of isoptics of strictly convex curves were studied in [4], [5], [9] and in [7] some results for not strictly convex curves can be found. Interesting extension of the notion of isoptic to non-euclidean spaces are given in [6].

In this paper we find an $z$-isoptic curve of a triangle for a fixed positive angle $z$, $z < \pi$ and study some of its properties. As an application of our results we show that a family of $z$-isoptics of properly chosen triangles has envelope which is $z$-isoptic of an oval.

2. Properties of an $\alpha$-isoptic curve of a triangle

Let $z$ be fixed and let $z_k = x_k + iy_k, \ k = 1, 2, 3$, denote the vertices of a counter-clockwise oriented triangle $T$ on the complex plane $\mathbb{C}$. If we ever use a subindex $k$ greater than 3, we always mean it modulo 3. For $k = 1, 2, 3$ we use the following notations. Let $\overline{a_k} = \frac{\overline{z_k + 1} \overline{z_{k+2}}}{a_k} = x_{k+2} - x_{k+1} + i(y_{k+2} - y_{k+1})$ be an oriented side of the triangle $T$, then $a_k = \sqrt{(x_{k+2} - x_{k+1})^2 + (y_{k+2} - y_{k+1})^2}$ denotes its length and $\beta_k$ is an angle of $T$ corresponding to the vertex $z_k$ and opposite to the side $\overline{a_k}$. Without loss of generality we can assume throughout this paper that $\beta_1 \geq \beta_2 \geq \beta_3$ or equivalently $a_1 \geq a_2 \geq a_3$.

From the inscribed angle theorem it is known that $C_z$, the $z$-isoptic curve of $T$ is a union of at least 3 and at most 6 circular arcs, one arc over each side of $T$ and if $z$ is greater then $\pi - \beta_k$ one has an additional arc over the vertex $z_k$. We find the equations for each part of $C_z$.

Over the side $\overline{a_k}$ a part of $C_z$ is an arc of a circle $C(s_k, r_k)$ centered at the point $s_k$ and with radius $r_k$. Moreover, if $l_k$ is the line containing the side $\overline{a_k}$, then the center $s_k$ and the vertex $z_k$ lie in the same half plane of
$l_k$ if and only if $\alpha$ is less than $\pi/2$. To find the equation of the circle $C(s_k, r_k)$ we observe that the point $z = x + iy$ belongs to the part $C_z$ over the side $\overrightarrow{a_k}$ if the angle between the vectors $\overrightarrow{u} = \overrightarrow{zz_{k+1}}$ and $\overrightarrow{v} = \overrightarrow{zz_{k+2}}$ is equal to $\pi - \alpha$. Using formula (1.1) we obtain the equation

$$|u| \cdot |v| \sin(\pi - \alpha) = (x - x_{k+1})(y - y_{k+2}) - (y - y_{k+1})(x - x_{k+2}).$$

Taking square of both sides of the above equation we get after some calculations

$$|z - \left(\frac{(z_{k+1} + z_{k+2})}{2} \pm i \cdot \cot \alpha \cdot (z_{k+1} - z_{k+2})\right)|^2 = \frac{a_k^2}{4 \sin^2 \alpha}.$$

(2.1)

Now, the property of the vertex $z_k$ and the center of the circle $s_k$ lying in the same half plane allow us to determine the sign in the above formula and finally we obtain

$$r_k = \frac{a_k}{2 \sin \alpha};$$

(2.2)

and

$$s_k = \frac{(z_{k+1} + z_{k+2})}{2} - i \cot \alpha \frac{(z_{k+1} - z_{k+2})}{2}.$$

(2.3)

Figure 1. Isoptics of $T$ for $\alpha \in \{\pi/12, 5\pi/24, 5\pi/12, 7\pi/12, 2\pi/3, 3\pi/4, 19\pi/24, 5\pi/6, 21\pi/24, 43\pi/48, 11\pi/12\}$. 
It is worth to notice that the arc of the circle $C(p_k, r_k)$ which is the part of $C_z$ over the vertex $z_k$, for $z$ less then $\pi - \beta_k$, is obtained in the same way. The only deference is that the center of the circle $p_k$ and vertex $z_k$ lie in the same half plane of $l_k$ if and only if $z$ is greater then $\pi/2$. Thus for $z$ less then $\pi - \beta_k$ from (2.1) we have the following

\[(2.4) \quad p_k = \frac{(z_{k+1} + z_{k+2})}{2} + i \cot z \frac{(z_{k+1} - z_{k+2})}{2}.
\]

**Example 2.1.** Let $T$ be a triangle with vertices $z_1 = 1$, $z_2 = i$, $z_3 = -i\sqrt{3}$. Then from formulas (2.1), (2.2), (2.3) and (2.4) we get the isoptics of $T$.

The most interesting question is for which $z$ the $z$-isoptic of $T$ is a convex curve. To study this problem we need to find the points of intersection of two circular arcs of $C_z$.

If $z$ is less or equal to $\pi - \beta_k$ then two circular arcs intersect at $z_k$. Let $z$ be greater then $\pi - \beta_k$. Let $\zeta_k$ denote the intersection point of the circles $C(s_{k+1}, r_{k+1})$, $C(p_k, r_k)$ and the line $l_{k+2}$, and let $\eta_k$ denote the intersection point of the circles $C(s_{k+2}, r_{k+2})$, $C(p_k, r_k)$ and the line $l_{k+1}$. We can find the coordinates of $\zeta_k$ and $\eta_k$ by straightforward calculations. To simplify notations we put $\zeta_k = \eta_k = z_k$ for $z \leq \pi - \beta_k$ and together we have

\[(2.5) \quad \zeta_k = z_k + (z_{k+1} - z_k) \frac{a_{k+1} \cdot \min\{0, \sin(z + \beta_k)\}}{a_{k+2} \sin z},
\]

\[(2.6) \quad \eta_k = z_k + (z_{k+2} - z_k) \frac{a_{k+2} \cdot \min\{0, \sin(z + \beta_k)\}}{a_{k+1} \sin z}.
\]

Now we state a useful property of $C_z$ which helps us to study its convexity.

**Proposition 2.2.** Let $z > \pi - \max\{\beta_1, \beta_2, \beta_3\}$ be fixed. Let $T$ be a given triangle in the complex plane and let $C_z$ be its $z$-isoptic curve. Then the triangle with the vertices $s_{k+1}, p_k, \zeta_k$ and the triangle with the vertices $p_k, s_{k+2}, \eta_k$ are geometrically congruent to each other and both are similar to $T$ for each $k \in \{1, 2, 3\}$ for which the arc of the circle $C(p_k, r_k)$ is a part of $C_z$.

**Proof.** Let $k \in \{1, 2, 3\}$ be arbitrarily chosen. If $z \leq \pi - \beta_k$, then none arc of the circle $C(p_k, r_k)$ is a part $C_z$. Let $z > \pi - \beta_k$. Then we
have $|\overrightarrow{s_{k+1}z_k}| = r_{k+1}$, $|\overrightarrow{s_{k+2}z_k}| = r_{k+2}$ and $|\overrightarrow{p_k\zeta_k}| = |\overrightarrow{p_k\eta_k}| = r_k$. We can obtain lengths $|\overrightarrow{s_{k+2}p_k}|$ and $|\overrightarrow{s_{k+1}p_k}|$ using formulas (2.3) and (2.4), and we get the following equality

$$|\overrightarrow{s_{k+2}p_k}| = \frac{(z_{k+2} - z_k)}{2} + i\cot\alpha\left(\frac{(z_{k+2} - z_k)}{2}\right) = r_{k+1},$$

and analogously $|\overrightarrow{s_{k+1}p_k}| = r_{k+2}$. Thus the triangle with the vertices $s_{k+1}, p_k, \zeta_k$ and the triangle with the vertices $p_k, s_{k+2}, \eta_k$ have the required properties. Since $k$ was chosen arbitrarily we get the proof. \hfill \Box

Proposition 2.2 allows us to reduce the domain of $\alpha$ to the interval $(0, \beta_1]$ while studying the convexity of $C_\alpha$. Namely we have

**Corollary 2.3.** Let $\alpha$ be fixed. If $\alpha > \pi - \max \{\beta_1, \beta_2, \beta_3\}$ then the $\alpha$-isoptic curve of $T$ is not convex.

**Proof.** Due to our assumption $\beta_1 = \max \{\beta_1, \beta_2, \beta_3\}$. Let $\alpha > \pi - \beta_1$ be fixed. Then, at least an arc of $C(p_1, r_1)$ is a part of $C_\alpha$. The vector $\overrightarrow{\zeta_1p_1}$ is normal to the tangent line to the circle $C(p_1, r_1)$ at the point $\zeta_1$ and the vector $\overrightarrow{\zeta_1s_2}$ is normal to the tangent line to the circle $C(s_2, r_2)$ at the point $\zeta_1$. By Proposition 2.2 and formula (1.1) we get that the circles $C(p_1, r_1)$ and $C(s_2, r_2)$ intersects at $\zeta_1$ under the angle $\pi - \beta_3$, thus $C_\alpha$ is not convex. \hfill \Box

In fact, the domain of convexity of the curve $C_\alpha$ is a proper subset of $(0, \beta_1]$. We prove the following

**Theorem 2.4.** Let $T$ be a given triangle in the complex plane and let $C_\alpha$ be its $\alpha$-isoptic curve. Then $C_\alpha$ is a convex curve if $\alpha \leq (\pi - \max \{\beta_1, \beta_2, \beta_3\})/2$.

**Proof.** By our assumption we have $\beta_1 = \max \{\beta_1, \beta_2, \beta_3\}$. Let $\alpha < \pi - \beta_1$ be fixed. Then $C_\alpha$ consists of 3 circular arcs $C(s_k, r_k), k = 1, 2, 3$. Let $\gamma_k$ denote the angle under which the circles $C(s_{k+1}, r_{k+1})$ and $C(s_{k+2}, r_{k+2})$ intersect at the point $z_k$. Similarly as in the proof of Corollary 2.3 the vector $\overrightarrow{z_k s_{k+2}}$ is normal to the tangent line to the circle $C(s_{k+2}, r_{k+2})$ at the point $z_k$ and the vector $\overrightarrow{z_k s_{k+1}}$ is normal to the tangent line to the circle $C(s_{k+1}, r_{k+1})$ at the point $z_k$. Using formula (1.1) we find
Let $\zeta_k s_{k+2}, \bar{\zeta}_k s_{k+1}$ and consequently $\gamma_k = \pi + 2\alpha + \beta_k$. Obviously $\gamma_k \leq 2\pi$ and hence we get the required result.

**Corollary 2.5.** The $\alpha$-isoptic curve of an equilateral polygon with $n$ sides is convex for $\alpha \leq \pi/n$ and the $(\pi/n)$-isoptic is a circle in which the polygon is inscribed.

**Proof.** If a polygon is an equilateral triangle then by Theorem 2.4 its $\alpha$-isoptic curve is convex for $\alpha \leq \pi/3$ and $C_{\pi/3}$ is a circle circumscribed on $T$.

Now let an equilateral polygon have $n$ sides, $n > 3$. By inscribed angle theorem and Theorem 2.4 its $\alpha$-isoptic curve is convex for $\alpha \leq \pi/n$. And again the $(\pi/n)$-isoptic is a circle.

3. The length of an $\alpha$-isoptic curve of a triangle

In this section we study some properties of the length function $L(\alpha)$ of $C_\alpha$ as a function of $\alpha$. To find the length function we need some additional notations. For $k = 1, 2, 3$ let $\varphi_k$ denote an angular measure in radians of the arc of the circle $C(s_k, r_k)$ which is a part of $C_\alpha$ and for a sufficiently large $\alpha$ let $\psi_k$ denote an angular measure in radians of the arc of the circle $C(p_k, r_k)$ which is also a part of $C_\alpha$. Then, by the inscribed angle theorem, we have for $k = 1, 2, 3$

\begin{align}
(3.1) \quad \varphi_k &= 2(\alpha - \max \{0, \beta_{k+1} + \alpha - \pi\} - \max \{0, \beta_{k+2} + \alpha - \pi\}), \\
(3.2) \quad \psi_k &= 2\max \{0, \beta_k + \alpha - \pi\}.
\end{align}

Using the above formulas we obtain the length of $C_\alpha$ as a function of $\alpha$

\begin{equation}
(3.3) \quad L(\alpha) = \sum_{k=1}^{3} (\varphi_k + \psi_k) r_k.
\end{equation}

**Theorem 3.1.** Let $T$ be a triangle in the complex plane and let $C_\alpha$ be an $\alpha$-isoptic curve of $T$ for a given angle $\alpha$. Then $L(\alpha)$ the length function of $C_\alpha$ defined by (3.3) is a continuous and strictly increasing function with respect to $\alpha$. The function $L$ is not convex.

**Proof.** Assume that $\beta_1 \geq \beta_2 \geq \beta_3$. Form (3.3) and the sine theorem we have the explicit formula for $L$
\[ L(z) = \begin{cases} 
\frac{2aR(\sin \beta_1 + \sin \beta_2 + \sin \beta_3)}{\sin z} & \text{for } 0 < z \leq \pi - \beta_1, \\
\frac{2R(2z \sin \beta_1 + (\pi - \beta_1)(-\sin \beta_1 + \sin \beta_2 + \sin \beta_3))}{\sin z} & \text{for } \pi - \beta_1 < z \leq \pi - \beta_2, \\
\frac{2R(z(\sin \beta_1 + \sin \beta_2 - \sin \beta_3) + (\beta_1 - \beta_2)(\sin \beta_1 - \sin \beta_2) + (\pi + \beta_3) \sin \beta_3)}{\sin z} & \text{for } \pi - \beta_2 < z \leq \pi - \beta_3, \\
\frac{4R(\beta_1 \sin \beta_1 + \beta_2 \sin \beta_2 + \beta_3 \sin \beta_3)}{\sin z} & \text{for } \pi - \beta_3 < z < \pi, 
\end{cases} \]

where \( R \) is the radius of the circle circumscribing \( T \). The straightforward calculations show that \( L \) is continuous at the points \( \pi - \beta_1, \pi - \beta_2, \pi - \beta_3 \) and thus at all \( z \in (0, \pi) \). Each function \( L_k(z), k = 1, 2, 3, 4 \) is differentiable in an open subset of its domain and the derivatives are equal to

\[ L_1'(z) = \frac{2R}{\sin^2 z} (\sin \beta_1 + \sin \beta_2 + \sin \beta_3)(\sin z - z \cos z), \]

\[ L_2'(z) = \frac{2R}{\sin^2 z} [2 \sin \beta_1(\sin z - z \cos z) - (\pi - \beta_1)(-\sin \beta_1 + \sin \beta_2 + \sin \beta_3) \cos z], \]

\[ L_3'(z) = \frac{2R}{\sin^2 z} \{ (\sin \beta_1 + \sin \beta_2 - \sin \beta_3)(\sin z - z \cos z) \\
- [(\beta_1 - \beta_2)(\sin \beta_1 - \sin \beta_2) + (\pi + \beta_3) \sin \beta_3] \cos z \}, \]

\[ L_4'(z) = \frac{-4R}{\sin^2 z} (\beta_1 \sin \beta_1 + \beta_2 \sin \beta_2 + \beta_3 \sin \beta_3) \cos z. \]

Since they have property

\[ (3.4) \quad L_{k+1}'(\pi - \beta_k) - L_k'(\pi - \beta_k) = \frac{a_k - a_{k+1} - a_{k+2}}{\sin \beta_k} < 0, \quad \text{for } k = 1, 2, 3, \]

thus the function \( L' \) is not defined at the points \( \pi - \beta_1, \pi - \beta_2, \pi - \beta_3 \).

We show that \( L' \) is a positive function in each interval of its domain and thus \( L \) is increasing.

First note that the function \( f(z) = \sin z - z \cos z \) is positive and increasing for \( z \in (0, \pi) \) and the function \( g(z) = -\cos z \) is positive for \( z \in (\pi/2, \pi) \). Consequently the functions \( L_1', L_3', L_4' \) are positive in their domains.
The function $L'_2$ is positive for $\alpha > \pi/2$ hence we have to show that it is positive in the interval $[\pi - \beta_1, \pi/2]$. Since the function
\[
h(\alpha) = 2 \sin \beta_1 (\sin \alpha - \alpha \cos \alpha) - (\pi - \beta_1)(- \sin \beta_1 + \sin \beta_2 + \sin \beta_3) \cos \alpha
\]
has nonnegative derivative for $\alpha \in [\pi - \beta_1, \pi/2]$ it is enough to show that $h(\pi - \beta_1)$ is positive. Let
\[
H(\beta_2, \beta_3) = h(\pi - \beta_1)
= 2 \sin^2 (\beta_2 + \beta_3) - (\beta_2 + \beta_3) \cos (\beta_2 + \beta_3)[\sin (\beta_2 + \beta_3) + \sin \beta_2 + \sin \beta_3].
\]
We need to show that the minimal value of $H$ in
\[
D = \{(\beta_2, \beta_3) \mid 0 \leq \beta_3 \leq \beta_2 \leq \pi/2, \ 0 \leq \beta_2 + \beta_3 \leq \pi/2\}
\]
is nonnegative. The function $H$ has no critical points inside $D$. Moreover,
\[
H(\beta_2, \pi/2 - \beta_2) = 2, \ \text{for} \ \beta_2 \in [\pi/4, \pi/2],
\]
\[
H(\beta_2, 0) = 2 \sin \beta_2 (\sin \beta_2 - \beta_2 \cos \beta_2) \geq 0, \ \text{for} \ \beta_2 \in [0, \pi/2].
\]
To complete this part of the proof we show that
\[
\tilde{H}(\beta_2) = H(\beta_2, \beta_2) = 2 \sin^2 2 - 2 \beta_2 \cos 2 \beta_2 (\sin 2 \beta_2 - 2 \sin \beta_2)
\]
is nonnegative for $\beta_2 \in [0, \pi/4]$. Once again, we use the fact that this function is nondecreasing and $\tilde{H}(0) = 0$. Indeed, we have
\[
\tilde{H}'(\beta_2) = 8 \cos \frac{\beta_2}{2} \bigg[ (3 \cos \beta_2 - 1) \sin \frac{\beta_2}{2} \cos 2 \beta_2 + \beta_2 (\cos 2 \beta_2 - 2 \cos 3 \beta_2) \cos \frac{\beta_2}{2} \bigg]
= 8 \cos \frac{\beta_2}{2} \bigg\{ \cos \frac{5\beta_2}{2} \sin^2 \frac{\beta_2}{2} (\beta_2 - \sin \beta_2) + \sin \frac{5\beta_2}{2} (3 \cos \beta_2 - 1) \sin^2 \frac{\beta_2}{2} \bigg\}
+ \bigg[ \cos \frac{5\beta_2}{2} (\sin \beta_2 \cos \beta_2 - \beta_2) + \frac{3}{2} \beta_2 \sin \beta_2 \sin \frac{5\beta_2}{2} \bigg].
\]
From the first formula we obtain that $\tilde{H}'(\beta_2)$ is positive for $\beta_2 \in (\pi/6, \pi/4]$. If $\beta_2 \in (0, \pi/6]$ each term in a square bracket in the second formula is positive. Thus, $\tilde{H}'(\beta_2)$ is positive for $\beta_2 \in (0, \pi/4]$. Finally, we get that $H(\beta_2, \beta_3)$ is nonnegative and is equal to 0 only if $\beta_2 = \beta_3 = 0$. This completes the proof that $L'_2$ is positive in its domain.

Moreover, the functions $L'_k, k = 1, 2, 3, 4$ are positive and the property (3.4) implies that the graph of $L$ is not convex. It is worth to notice that each $L_k, k = 1, 2, 3, 4$ is convex in its domain since $(f(\alpha)/\sin^2 \alpha)' = \alpha (1 + \cos^2 \alpha)/\sin^3 \alpha > 0$ and $(- g(\alpha)/\sin^2 \alpha)' = (1 + \cos^2 \alpha)/\sin^3 \alpha > 0$ for $\alpha \in (0, \pi)$. □
Example 3.2. For the triangle \( T \) from Example 2.1 the length function \( L(x) \) given by (3.3) has the graph shown on Figure 2.

![Graph of the length function \( L(x) \) of \( T \).](image)

4. The area of an \( \alpha \)-isoptic curve of a triangle

Our aim in this section is to investigate some properties of the area function \( A(x) \) of the \( z \)-isoptic curve of \( T \) as a function of \( \alpha \). Let \( \zeta_1 \eta_1 \zeta_2 \eta_2, \zeta_3 \eta_3 \), where \( \zeta \) and \( \eta \) are defined by (2.5) and (2.6), respectively, be a counterclockwise oriented polygon. If \( \eta_k = \zeta_k = z_k \) then the point \( z_k \) is counted only once in the polygon. Using formulas (3.1) and (3.2) we get the area function of \( C_x \)

\[
A(x) = \text{area of } \zeta_1 \eta_1 \zeta_2 \eta_2 \zeta_3 \eta_3 \\
+ \sum_{k=1}^{3} \left( \varphi_k r_k^2 - \frac{1}{2} r_k^2 \sin \varphi_k \right) + \sum_{k=1}^{3} \left( \psi_k t_k^2 - \frac{1}{2} t_k^2 \sin \psi_k \right).
\]

The behavior of the function \( A(x) \) is described in the following

Theorem 4.1. Let \( T \) be a triangle in the complex plane and let \( C_x \) be an \( z \)-isoptic curve of \( T \) for a given angle \( z \). Then \( A(x) \), the area function of \( C_x \) defined by (4.1) is a differentiable, strictly increasing and convex function with respect to \( x \).
\textbf{Proof.} Assume that }β_1 ≥ β_2 ≥ β_3 \text{ and let } ζ_k \text{ and } η_k \text{ be defined by (2.5) and (2.6), respectively. To compute the area function given by (4.1) we need to find the area of a polygon } ζ_1 η_1 ζ_2 η_2 ζ_3 η_3 \text{. To this end we use theorem 3 given by Radić in [10]. He proved that

The area of } ζ_1 η_1 ζ_2 η_2 ζ_3 η_3 = \frac{1}{2} |ζ_1 + η_1 + ζ_2, η_2 + ζ_3, η_3 + ζ_3 + η_3 + ζ_1|,

\text{where}

|z_1 + z_2, z_2 + z_3, \ldots, z_n + z_1| = \sum_{1 \leq i < j \leq n} (-1)^{3+i+j} [z_i, z_j].

\text{Obviously, if } η_k = ζ_k = z_k \text{ then the point } z_k \text{ is counted only once in the polygon. Applying the sine theorem we get

\begin{align*}
A(α) &= \begin{cases} 
2R^2[(x - \sin x \cos x)(1 + \cos β_1 \cos β_2 \cos β_3) + \sin^2 x \sin β_1 \sin β_2 \sin β_3] \\
\sin^2 x & \text{for } 0 < α ≤ π - β_1,
\end{cases} \\
&= \begin{cases} 
2R^2[(x - \sin x \cos x)\sin^2 β_1 + \sin β_2 \sin β_3[(π - β_1) \cos β_1 + \sin β_1]] \\
\sin^2 x & \text{for } π - β_1 < α ≤ π - β_2,
\end{cases} \\
&= \begin{cases} 
2R^2[(x \cos β_1 + \cos x \sin(β_1 - x)) \sin β_1 \sin β_2 + (β_1 + \beta_2) \cos β_1 + \sin β_1] \sin β_2 \sin β_3] \\
\sin^2 x & \text{for } π - β_2 < α ≤ π - β_3,
\end{cases} \\
&= \begin{cases} 
\frac{2R^2(π - β_2)\sin^2 β_3}{\sin^2 x} & \text{for } π - β_3 < α < π,
\end{cases}
\end{align*}

\text{where } R \text{ is the radius of the circle circumscribed on } T. \text{ The function } A(α) \text{ is continuous at the points } π - β_1, π - β_2, π - β_3 \text{ and thus at all } α \in (0, π). \text{ The same is true for its derivative and we have

\begin{align*}
A'_1(α) &= \frac{4R^2}{\sin^3 x} \{(\sin x - α \cos x)(1 + \cos β_1 \cos β_2 \cos β_3) - \sin^2 β_1 \sin β_2 \sin β_3[(π - β_1) \cos β_1 + \sin β_1]\},
\end{align*}

\begin{align*}
A'_2(α) &= \frac{4R^2}{\sin^3 x} \{(\sin x - α \cos x) \sin^2 β_1 \\
&- \cos x \sin β_2 \sin β_3[(π - β_1) \cos β_1 + \sin β_1]\},
\end{align*}
\[ A_3'(\alpha) = \frac{4R^2}{\sin^3 x} \left\{ [\sin(\alpha - \beta_3) - \alpha \cos \alpha \cos \beta_3] \sin \beta_1 \sin \beta_2 \\
- \cos \alpha [(\pi - \beta_2) \sin \beta_1 \cos \beta_2 \sin \beta_3 + ((\pi - \beta_1) \cos \beta_1 + \sin \beta_1) \sin \beta_2 \sin \beta_3]\right\}, \]

\[ A_4'(\alpha) = \frac{-4R^2}{\sin^3 x} \cos \alpha [3 \sin \beta_1 \sin \beta_2 \sin \beta_3 + \beta_1 \sin^2 \beta_1 + \beta_2 \sin^2 \beta_2 + \beta_3 \sin^2 \beta_3]. \]

where the derivative of the function \( A_k(\alpha), \ k = 1, 2, 3, 4, \) is defined in its domain. Using functions \( f \) and \( g \) defined in the proof of Theorem 3.1 we immediately obtain that \( A_1', A_3', A_4' \) are positive in their domains. Moreover, since \( f'(\alpha) \geq f'(\pi - \beta_1) \) we have

\[ A_2'(\alpha) \geq \frac{2f(\alpha)(\sin^2 \beta_1 - \cos \alpha \sin \beta_2 \sin \beta_3)}{\sin^3 x} \geq \frac{2f(\alpha)(\sin^2 \beta_1 - \sin \beta_2 \sin \beta_3)}{\sin^3 x} > 0. \]

Since \( A'(\alpha) \) is positive then the function \( A(\alpha) \) is strictly increasing.

The second derivative of the function \( A_k(\alpha), \ k = 1, 2, 3, 4, \) is defined in each open subset of its domain and it is equal to

\[ A_1''(\alpha) = \frac{4R^2}{\sin^4 x} [\alpha(1 + 2 \cos^2 \alpha) - 3 \sin \alpha \cos \alpha](1 + \cos \beta_1 \cos \beta_2 \cos \beta_3), \]

\[ A_2''(\alpha) = \frac{4R^2}{\sin^4 x} \left\{ [\alpha(1 + 2 \cos^2 \alpha) - 3 \sin \alpha \cos \alpha] \sin^2 \beta_1 \\
+ (1 + 2 \cos^2 \alpha)[(\pi - \beta_1) \cos \beta_1 + \sin \beta_1] \sin \beta_2 \sin \beta_3\right\}, \]

\[ A_3''(\alpha) = \frac{4R^2}{\sin^4 x} \left\{ [\alpha(1 + 2 \cos^2 \alpha) - 3 \sin \alpha \cos \alpha] \sin \beta_1 \sin \beta_2 \cos \beta_3 \\
+ (1 + 2 \cos^2 \alpha)[(\pi - \beta_2) \sin \beta_1 \cos \beta_2 \sin \beta_3 \\
+ ((\pi - \beta_1) \cos \beta_1 + \sin \beta_1) \sin \beta_2 \sin \beta_3 + \sin \beta_1 \sin \beta_2 \sin \beta_3]\right\}, \]

\[ A_4''(\alpha) = \frac{4R^2}{\sin^4 x} (1 + 2 \cos^2 \alpha)[3 \sin \beta_1 \sin \beta_2 \sin \beta_3 + \beta_1 \sin^2 \beta_1 \\
+ \beta_2 \sin^2 \beta_2 + \beta_3 \sin^2 \beta_3]. \]
Thus the function $A''(x)$ is not continuous at the points $\pi - \beta_1$, $\pi - \beta_2$, $\pi - \beta_3$, but still, it is positive in its domain. This completes the proof.

**Example 4.2.** Once again, let us take the triangle $T$ from Example 2.1. Then the graph of the area function $A(x)$ given by (4.1) is shown on Figure 3.

![Figure 3. Graph of the area function $A(x)$ of $T$.](image)

5. Application of $x$-isoptic curves of a triangle

Let $x \in (0, \pi)$ be fixed. In this section we use results obtained in Section 2 to study the $x$-isoptic $C_x$ of an oval $C$. By an oval we understand $C^2$, plane closed simple curve with positive curvature.

Let $p(t) \in C^2([0, 2\pi]), \ t \in [0, 2\pi]$ be the support function of $C$. Then the point $z_2(t)$ satisfying (1.3) belongs to $C_x$ and it is an intersection of two lines tangent to $C$ at points $z(t)$ and $z(t + x)$. Let $\xi$ be an arbitrary point from an open angle $\angle (z_2(t)z(t + x), z_2(t)z(t))$ and let $T$ be a counter-clockwise oriented triangle with vertices $z(t + x)$, $z(t)$, $\xi$. By $C_{x,t}$ we denote an arc of a circle

$$
(5.1) \quad z - \frac{z(t + x) + z(t) + i \cot x (z(t + x) - z(t))}{2} = \frac{|z(t + x) - z(t)|}{2 \sin x}
$$

which is also a part of $x$-isoptic of $T$. We should mention that $C_{x,t}$ does not depend on $\xi$. If $[\xi z(t + x), \xi z(t)] > 0$ then the center of the circle in (5.1) is
obtained from equation (2.4), overwise it is obtained from equation (2.3). 
Finally we define the family of arcs as follows 
\[ (5.2) \quad \mathcal{F}_z = \{ C_{x,t}, t \in [0, 2\pi] \}. \]

Using the above notations we have

**Theorem 5.1.** \( C_z \) is the envelope of the family \( \mathcal{F}_z \) defined by (5.2).

**Proof.** Let \( F(x, y, t) = 0 \) denote an equation for the family \( \mathcal{F}_z \) given by (5.2). Then, applying formula (1.2) to (5.1) with \( z = x + iy \), we get
\[ (5.3) \quad F(x, y, t) = (x^2 + y^2) \sin z \]
\[ + x[p(t + z) \sin t + \dot{p}(t + z) \cos t - p(t) \sin (t + z) - \dot{p}(t) \cos (t + z)] \]
\[ + y[-p(t + z) \cos t + \dot{p}(t + z) \sin t + p(t) \cos (t + z) - \dot{p}(t) \sin (t + x)] \]
\[ + p(t + z)\dot{p}(t) - p(t)\dot{p}(t + x) = 0, \]

thus \( \mathcal{F}_z \) is indeed a one parameter family of arcs. Moreover, the point
\[ z_x(t) = x_z(t) + iy_z(t) \]
\[ = \frac{p(t) \sin (t + x) - p(t + x) \sin t + i(-p(t) \cos (t + x) + p(t + x) \cos t)}{\sin z} \]
defined by (1.3) satisfies the equation (5.3) and thus \( z_x(t) \in C_{x,t} \) since \( \xi \) is an interior point of the angle \( \angle(z_x(t), z(t), z_x(t)) \).

Now Theorem 4 in [1] asserts that \( C_{x,t} \) and \( C_z \) are tangent at the point \( z_x(t) \). To show that \( C_z \) is the envelope of \( \mathcal{F}_z \) it is enough to check that \( F'_t(x, y, t) = 0 \) at \( z_x(t) \) (see, e.g., [3] or [11]). Indeed, we have
\[ F'_t(x, y, t) = x[R(t + x) \cos t - R(t) \cos (t + x)] \]
\[ + y[R(t + x) \sin t + R(t) \sin (t + x)] + p(t + x)R(t) - p(t)R(t + x), \]
where \( R(t) = p(t) + \dot{p}(t) \) is a radius of curvature of \( C \), and finally,
\[ F'_t(x_z(t), y_z(t), t) = 0, \]
which completes the proof. \( \square \)

Theorem 5.1 remains true in special case when \( T \) is inscribed in oval \( C \).

The above considerations can be related to those in paper of Martini [8] on the classical light field theory in \( \mathbb{R}^d \), \( d \geq 2 \).
REFERENCES


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