Peierls’ substitution for low lying spectral energy windows

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Abstract. We consider a $2d$ periodic magnetic Schrödinger operator perturbed by a weak magnetic field which slowly varies around a positive mean. In a previous paper we proved the appearance of a “Landau type” structure of spectral islands at the bottom of the spectrum, under the hypothesis that the lowest Bloch eigenvalue of the unperturbed operator remained simple on the whole Brillouin zone, even though its range may overlap with the range of the second eigenvalue. We also assumed that the first Bloch spectral projection was smooth and had a zero Chern number.

In this paper we extend our previous results to the only two remaining possibilities: either the first Bloch eigenvalue remains isolated while its corresponding spectral projection has a non-zero Chern number, or the first two Bloch eigenvalues cross each other.

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Contents

1 Introduction ................................................................. 2
2 The main results .......................................................... 8
3 The quasi Wannier system ............................................... 13
4 An abstract reduction argument ...................................... 15
5 The magnetic pseudodifferential calculus ........................... 19
6 The magnetic quasi-band projections ................................. 23
7 Checking the conditions of the abstract reduction procedure for the magnetic operators .................................................. 25
8 The one band effective Hamiltonian ................................. 27
References ................................................................. 42

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1. Introduction

1.1. Some history of the problem and the structure of the paper. Since the pioneering work of Peierls [27] and Luttinger [21], physicists consider Peierls’ substitution to be the right way of taking into account external magnetic fields in periodic solid state systems. Very roughly, the receipt is as follows: consider a non-magnetic periodic Schrödinger Hamiltonian having an energy band \( \lambda_0(\theta) \) which is periodic with respect to the Brillouin zone; then in the presence of an external magnetic vector potential \( A \), the right physical behaviour of the magnetically perturbed system should be given by \( \lambda_0(\theta - A(x)) \), whatever that means.

From a rigorous mathematical point of view the situation is more complicated, and here we will only discuss the case in which the external magnetic perturbation is weak and slowly variable. The smallness of the magnetic perturbation is generically represented by a parameter \( \epsilon \geq 0 \), where \( \epsilon = 0 \) corresponds to the unperturbed objects. The first natural question one asks in this context is whether one can construct an effective Hamiltonian which is close in some sense to the original one, either dynamically or spectrally. The second question is to analyse the effective model and draw conclusions on its spectral and dynamical properties.

To the best of our knowledge, all the effective models constructed until now suppose that the unperturbed Bloch spectrum under investigation remains isolated from the rest; this does not require a gap in the full spectrum, only a non-empty gap at each fixed quasi-momentum \( \theta \) of the Brillouin zone (the non-crossing, range-overlapping case). On top of that, with the notable exception of a recent work by Freund and Teufel [11], all existing papers require that the Bloch projection associated to the Bloch spectrum under investigation is topologically trivial. The triviality condition leads to the existence of a basis consisting of spatially localized Wannier functions [5, 7, 10, 15, 24, 25] which plays an essential role in the reduction procedure. We note that although [11] can treat non-trivial projections, only bounded perturbations are allowed.

When the external magnetic field is constant and the range of \( \lambda_0 \) does not overlap with the other ones (the gapped case), one can construct a smooth and periodic symbol \( \lambda_\epsilon(\xi) \) such that the Weyl quantization of \( \lambda_\epsilon(\xi - A_\epsilon(x)) \) is isospectral with the “true” magnetic band Hamiltonian, provided \( \epsilon \) is small enough. This is essentially due to Nenciu and Helffer and Sjöstrand [15, 25, 30]. Concerning the spectral analysis of these effective magnetic pseudo-differential operators there exists a large amount of literature which classifies the spectral gaps induced by the magnetic field; see the works by Bellissard [2, 3], Bellissard and Rammal [28], Helffer and Sjöstrand [14, 15] and references therein.
When the magnetic perturbation comes from a non-constant but slowly variable magnetic field, still under the assumption that a localized Wannier basis exists, one can again construct an effective Hamiltonian up to any order in $\epsilon$ (see [6, 9, 26]) but which typically lives in an $\epsilon$-dependent subspace. Thus the spectral analysis of the effective operator seems to be as complicated as the original one.

In our previous paper [4] we proved (for the first time in the non-constant case) the appearance of a “Landau type” structure at the bottom of the spectrum consisting of spectral islands of width of order $\epsilon$, separated by gaps of roughly the same order of magnitude; in [4] we still worked under the hypothesis that the lowest Bloch eigenvalue of the unperturbed operator remained simple on the whole Brillouin zone and a localized Wannier basis can be constructed.

The current manuscript extends our previous results to the only two remaining possibilities: either the first Bloch eigenvalue remains isolated while its corresponding spectral projection has a non-zero Chern number, or the first two Bloch eigenvalues cross each other.

Our main result is Theorem 2.2. It roughly states that in a certain narrow energy window near the bottom of the spectrum of the “true” perturbed Hamiltonian, the perturbed spectrum is well approximated by the spectrum of a pseudo-differential operator corresponding to a magnetically quantized periodic symbol which only depends on $\xi$. This result is new also for the constant magnetic case since we no longer need global non-crossing and/or triviality conditions on the lowest Bloch eigenvalue.

Another important consequence is that we can prove that gaps of order $\epsilon$ open near the bottom of the perturbed spectrum, due to the fact that the spectrum of our effective Hamiltonian has the same property. For more details see corollaries 2.3 and 2.4.

The paper is organized as follows.

- In the rest of this section we introduce some of the objects we want to study and we briefly review the Bloch–Floquet theory.

- In Section 2 we state our main results (Theorem 2.2 and corollaries 2.3 and 2.4). In subsection 2.3 we outline the main ideas behind the proofs.

- In Section 3 we construct a Wannier-like localized basis which spans a certain energy window of the unperturbed Hamiltonian; in this window, the lowest Bloch eigenvalue has to be non-degenerate.

- In Section 4 we present a general reduction argument based on the Feshbach–Schur map which permits the construction of an effective Hamiltonian in a narrow window near the bottom of the spectrum. Let us point out here
that our generalization of the usual Feshbach–Schur argument differs in spirit of some other interesting generalizations existing in the literature ([1, 12]). While still working with “sharp” projections, we no longer use the smallness of the commutator of this projection with the Hamiltonian but use instead the smallness of the spectral parameter and a second order expansion of the resolvent that obliges us to a “dressing” operation on the projected Hamiltonian.

- In Section 5 we review the so-called magnetic pseudo-differential calculus where the symbol composition rule is a twisted Moyal product, while in Section 6 we construct the magnetic counterpart of the Wannier-like projection from Section 3.

- In Section 7 we show that the magnetic Hamiltonian satisfies the conditions of the abstract reduction procedure which we developed in Section 4.

- In Section 8 we analyse the structure of the effective magnetic matrix constructed in Section 7. In particular, we show in Proposition 8.12 that the spectrum of this magnetic matrix is close to the spectrum of a magnetic pseudo-differential operator whose symbol $\tilde{\xi}(\xi)$ is smooth and periodic in $\xi$ while independent of $x$. In particular, this completes the proof of Theorem 2.2(i). In Proposition 8.13 we show that the function $\tilde{\xi}(\cdot)$ is close to the unperturbed Bloch eigenvalue $\lambda_0$ near its minimum; this ends the proof of Theorem 2.2. Finally, we prove corollaries 2.3 and 2.4.

### 1.2. Notation and conventions.

For any real finite dimensional Euclidean space $V$ we shall denote by

1. $BC^\infty(V)$ (respectively $C^\infty_{\text{pol}}(V)$) the family of smooth complex functions on $V$ that are bounded (respectively polynomially bounded) together with all their derivatives,

2. $\tau_v$ the translation by $-v \in V$ on any class of distributions on $V$,

3. the “Japanese bracket” is denoted by $\langle v \rangle := \sqrt{1 + |v|^2}$ for any $v \in V$ with $|v| \in \mathbb{R}_+$ its Euclidean norm,

4. $S(V)$ the space of Schwartz functions on $V$ and with $S'(V)$ the space of tempered distributions on $V$.

For any Banach space $B$ we denote by $\mathcal{L}(B)$ the Banach space of continuous linear operators in $B$. For any Hilbert space $\mathcal{H}$ and any pair of unit vectors $(u, v)$ in it we use the physics notation $|u\rangle\langle v|$ for the the projector

$$\mathcal{H} \ni w \mapsto |u\rangle\langle v| w := \langle v, w \rangle \mathcal{H} u.$$ (1.1)
Our scalar products are anti-linear in the first factor. As we are working in a two-dimensional framework, we use the following notation for the vector product of two vectors $u$ and $v$ in $\mathbb{R}^2$:

$$u \wedge v := u_1v_2 - u_2v_1.$$  \hfill (1.2)

and the $-\pi/2$ rotation of $v$:

$$v^\perp := (v_2, -v_1).$$  \hfill (1.3)

1.3. The Hamiltonian. Let $\Gamma \subset \mathbb{R}^2 \equiv \mathcal{X}$ be a regular periodic lattice. We consider a smooth $\Gamma$-periodic potential $V_{\Gamma}: \mathcal{X} \to \mathbb{R}$ and a smooth zero flux $\Gamma$-periodic magnetic field $B_{\Gamma}: \mathcal{X} \to \mathbb{R}$. A magnetic flux which is a rational multiple of $2\pi$ could also be considered but the Floquet reduction must involve magnetic translations. Moreover, in the non-zero flux case one has to work on a sub-lattice with symbols taking values in a finite dimensional complex vector space, and the spectral problem becomes qualitatively different. We plan to consider this extension in a future paper, while here we only deal with the zero flux case where a periodic magnetic potential can be constructed. Let us denote by $A_{\Gamma}: \mathcal{X} \to \mathbb{R}^2$ such a smooth $\Gamma$-periodic vector potential generating the magnetic field $B_{\Gamma}$.

We consider the Hilbert space $\mathcal{H} := L^2(\mathcal{X})$ and define the periodic Schrödinger operator

$$H^0 := (-i \partial_{x_1} - A^1_{\Gamma})^2 + (-i \partial_{x_2} - A^2_{\Gamma})^2 + V_{\Gamma},$$  \hfill (1.4)

as the unique self-adjoint extension in $\mathcal{H}$ of the above differential operator initially defined on $S(\mathcal{V})$ (see [29]).

As a perturbation of the periodic case, we consider a family of magnetic fields indexed by $(\epsilon, \kappa) \in [0, 1] \times [0, 1]$:

$$B^{\epsilon,\kappa}(x) := \epsilon B_0 + \kappa \epsilon B(\epsilon x).$$  \hfill (1.5)

Here $B_0 > 0$ is constant, while $B: \mathcal{X} \to \mathbb{R}$ is of class $BC^\infty(\mathcal{X})$. We choose the vector potential $A^0$ of the constant magnetic field $B_0$ in the form (transverse gauge)

$$A^0(x) = (B_0/2)(-x_2, x_1)$$  \hfill (1.6)

and some vector potential $A: \mathcal{X} \to \mathbb{R}^2$, that we can choose to be of class $C^\infty_{\text{pol}}(\mathcal{X})$ such that

$$B_0 = \partial_1 A^0_2 - \partial_2 A^0_1, \quad B = \partial_1 A_2 - \partial_2 A_1.$$  \hfill (1.7)
Similarly, for

\[ A^{\epsilon, \kappa}(x) := \epsilon A^0(x) + \kappa A(\epsilon x), \]

we have

\[ B^{\epsilon, \kappa} = \partial_1 A^{\epsilon, \kappa}_2 - \partial_2 A^{\epsilon, \kappa}_1. \]  

(1.8)

We shall also use the notation

\[ B^{\epsilon} := \epsilon B_0, \quad A^{\epsilon} := \epsilon A_0, \quad B_\epsilon(x) := B(\epsilon x), \quad A_\epsilon(x) := A(\epsilon x). \]  

(1.9)

**Definition 1.1.** We consider the magnetic Schrödinger operators defined as the self-adjoint extension in \( \mathcal{H} \) of the following differential operators:

\[ H^{\epsilon, \kappa} := (-i \partial_{x_1} - A_1^\Gamma - A_1^{\epsilon, \kappa})^2 + (-i \partial_{x_2} - A_2^\Gamma - A_2^{\epsilon, \kappa})^2 + V_\Gamma. \]  

(1.10)

\[ H^{\epsilon} := H^{\epsilon, 0} = (-i \partial_{x_1} - A_1^\Gamma - A_1^{\epsilon})^2 + (-i \partial_{x_2} - A_2^\Gamma - A_2^{\epsilon})^2 + V_\Gamma. \]  

(1.11)

When \( \kappa = \epsilon = 0 \), we recover \( H^0 \).

**1.4. The Bloch–Floquet theory.** We consider the quotient space \( \mathbb{R}^2 / \Gamma \) which is canonically isomorphic to the two dimensional torus \( \mathbb{T}^2 \equiv \Gamma \) and denote by \( \mathbb{R}^2 \ni x \mapsto \hat{x} \in \Gamma \) the canonical projection onto the quotient. We choose two generators \( \{e_1, e_2\} \) for the lattice \( \Gamma \) and the associate elementary cell

\[ E_\Gamma = \left\{ y = \sum_{j=1}^{2} t_j e_j \in \mathbb{R}^2 \left| \begin{array}{c} \text{for all } j \in \{1, 2\} \end{array} \right. \right\}, \]  

so that we have a bijection \( \mathcal{X} \ni x \mapsto (\gamma(x), y(x)) \in \Gamma \times E_\Gamma \). The dual lattice of \( \Gamma \) is then defined as

\[ \Gamma^* := \{ \gamma^* \in \mathcal{X}^* \mid \langle \gamma^*, \gamma \rangle \in 2\pi \mathbb{Z}, \text{ for all } \gamma \in \Gamma \}. \]

Considering the dual basis \( \{e_1^*, e_2^*\} \subset \mathcal{X}^* \) of \( \{e_1, e_2\} \), which is defined by

\[ \langle e_j^*, e_k \rangle = 2\pi \delta_{j,k}, \text{ we have } \Gamma^* := \bigoplus_{j=1,2} \mathbb{Z} e_j^*. \]

We similarly define \( T^* := \mathcal{X}^*/\Gamma^* \) and \( E_{T^*} := E_\Gamma \). We shall use \( \theta \) for the generic variable on the dual torus \( T^* \) and keep \( \xi \) for the momentum variable in the dual of the configuration space \( \mathcal{X}^* \).

We recall that the Floquet unitary map \( \mathcal{U}_\Gamma \) is defined by

\[ \mathcal{S}(\mathcal{X}) \ni \phi \mapsto \mathcal{U}_\Gamma \phi \in C^\infty(\mathcal{X} \times T^*_\Gamma), \quad \text{with } [\mathcal{U}_\Gamma \phi](x, \theta) := \sum_{\gamma \in \Gamma} e^{i \langle \gamma, \theta \rangle} \phi(x - \gamma), \]  

(1.13)

so that we have the following property

\[ [\mathcal{U}_\Gamma \phi](x + \gamma, \theta) = e^{i \langle \gamma, \theta \rangle} [\mathcal{U}_\Gamma \phi](x, \theta), \quad \text{for all } \gamma \in \Gamma, (x, \theta) \in \mathcal{X} \times T^*_. \]  

(1.14)
For any \( \theta \in \mathbb{T}_* \) we introduce
\[
\mathcal{F}_\theta := \{ v \in L^2_{\text{loc}}(\mathbb{X}) \mid v(x + \gamma) = e^{i(\theta, \gamma)}v(x), \text{ for all } \gamma \in \Gamma \} \tag{1.15}
\]
and notice that it is a Hilbert space for the scalar product
\[
\langle v, w \rangle_{\mathcal{F}_\theta} := \int_E \bar{v}(x)w(x)dx. \tag{1.16}
\]

For each \( \theta \in \mathbb{T}_* \), we define \( \mathcal{V}_\theta : \mathcal{F}_\theta \rightarrow L^2(\mathbb{T}) \) as the multiplication with the function \( \sigma_\theta(x) := e^{-i(\theta, x)} \). We can then form the associated direct integral of Hilbert spaces
\[
\mathcal{F} := \int_{\mathbb{T}_*} \mathcal{F}_\theta d\theta, \tag{1.17}
\]
and the following unitary map from \( \int_{\mathbb{T}_*} \mathcal{F}_\theta d\theta \) onto \( L^2(E_*) \otimes L^2(\mathbb{T}) \) defined by
\[
\mathcal{V}_{\mathbb{T}_*} := \int_{\mathbb{T}_*} \mathcal{V}_\theta d\theta. \tag{1.18}
\]

Then \( \mathcal{U}_\Gamma \) defines a unitary operator \( L^2(\mathbb{X}) \rightarrow \mathcal{F} \).

Note that a bounded operator \( X \in \mathcal{L}(\mathcal{H}) \) commutes with all the translations with elements from \( \Gamma \) if and only if there exists a measurable family \( \mathbb{T}_* \ni \theta \mapsto \hat{X}(\theta) \in \mathcal{L}(\mathcal{F}_\theta) \) such that
\[
\mathcal{U}_\Gamma X \mathcal{U}_\Gamma^{-1} = \int_{\mathbb{T}_*} \hat{X}(\theta)d\theta.
\]

The following statements are well known (see for example [20, 29]).

1. We have the direct integral decomposition
\[
\hat{H}^0 := \mathcal{U}_\Gamma H^0 \mathcal{U}_\Gamma^{-1} = \int_{\mathbb{T}_*} \hat{H}^0(\theta)d\theta, \tag{1.19}
\]
with
\[
\hat{H}^0(\theta) := (-i\partial_{x_1} - A_1^\Gamma)^2 + (-i\partial_{x_2} - A_2^\Gamma)^2 + V_\Gamma,
\]
whose domain in \( \mathcal{F}_\theta \) is the local Sobolev space
\[
\mathcal{H}_{\theta, 0}^2 := \{ v \in \mathcal{H}_{\text{loc}}^2(\mathbb{X}) \mid v(x + \gamma) = e^{i(\theta, \gamma)}v(x), \text{ for all } \gamma \in \Gamma \} \subset \mathcal{F}_\theta. \tag{1.20}
\]
(2) The family $\tilde{H}^0(\theta) (\theta \in \mathbb{T}_*)$ is unitarily equivalent with an analytic family
of type A in the sense of Kato [18]. The unitary operator is given by the
multiplication with $\mathcal{V}_0$ which maps $\mathcal{F}_\theta$ to $L^2(\mathbb{T})$. The rotated operator is
essentially self-adjoint on the set of smooth and periodic functions in $L^2(E)$
and acts on them as
\[ \tilde{H}^0(\theta) := (-i \nabla - A^\Gamma + \theta)^2 + V_\Gamma, \quad \theta \in E_* . \]

(3) There exists a family of continuous functions $\mathbb{T}_* \ni \theta \mapsto \lambda_j(\theta) \in \mathbb{R}$ indexed
by $j \in \mathbb{N}$, called the Bloch eigenvalues, such that $\lambda_j(\theta) \leq \lambda_{j+1}(\theta)$ for every
$j \in \mathbb{N}$ and $\theta \in \mathbb{T}_*$, and
\[
\sigma(\tilde{H}^0(\theta)) = \bigcup_{j \in \mathbb{N}} \{ \lambda_j(\theta) \} = \sigma(\tilde{H}^0(\theta)). \tag{1.21}
\]

(4) For each fixed $\theta$ we can define the Riesz spectral projections using complex
Cauchy integrals of the resolvent $(z - \tilde{H}^0(\theta))^{-1}$ around each eigenvalue $\lambda_j(\theta)$. If the lowest eigenvalue $\lambda_0(\theta)$ is always non-degenerate, it is also
smooth in $\theta$ and its corresponding Riesz projection $\hat{\pi}_0(\theta)$ is globally smooth
on the dual torus and has rank one. If $\lambda_0$ can cross with the other higher
eigenvalues, then by convention, at these crossing points $\hat{\pi}_0(\theta)$ is defined
as the higher-rank Riesz projection which encircles the still lowest, but now
degenerate eigenvalue. In this way, the projection $\hat{\pi}_0(\theta)$ is well defined on
the whole dual torus but it is no longer continuous.

2. The main results

2.1. (Non)existence of localized Wannier functions. In our previous work [4]
we imposed the condition that $\lambda_0$ had to remain non-degenerate on the whole
dual torus, while the unperturbed periodic operator $H^0$ had to be time-reversal
invariant (i.e. commuting with the complex conjugation operator). Under these
conditions, the orthogonal projection family $\hat{\pi}_0(\theta)$ is smooth and time-reversal
symmetric in the sense that
\[ C \hat{\pi}_0(\theta) = \hat{\pi}_0(-\theta) C \]
where $C$ is the anti-unitary operator given by the complex conjugation. In partic-
ular, this implies that the Chern number of this family equals zero and that one
can construct [15, 25, 10, 7] a global smooth section $\hat{\phi}_0: \mathbb{T}_* \to \mathcal{F}$ such that
\[ \hat{\pi}_0(\theta) = |\hat{\phi}_0(\theta)\rangle \langle \hat{\phi}_0(\theta)|. \tag{2.1} \]
Thus we have the orthogonal projection in \( \mathcal{H} \)
\[
\pi_0 := \mathcal{U}_\Gamma^{-1} \hat{\pi}_0 \mathcal{U}_\Gamma,
\]
which commutes with \( H^0 \) but is not necessarily a spectral projection of it (unless the range of \( \lambda_0 \) is isolated from the rest of the spectrum). When a smooth global section \( \phi_0 \) exists, the principal Wannier function \( \phi_0 \) is defined by
\[
\phi_0(x) := |\mathcal{U}^{-1}_\Gamma \hat{\phi}_0|(x) = |E_*|^{-\frac{1}{2}} \int_{T_*} \hat{\phi}_0(x, \theta) d\theta
\]
and has rapid decay. The family \( \phi_y := \tau_y \phi_0 \) generated from \( \phi_0 \) by translations over the lattice \( \Gamma \) is an orthonormal basis for the subspace \( \pi_0 \mathcal{H} \), consisting of (exponentially) localized functions.

As we have already mentioned, the main spectral result in [4] was obtained under the hypothesis that a global smooth section as in (2.1) exists, or equivalently, that there exists a localized Wannier basis for \( \pi_0 \). Our main concern in the present paper is to show that roughly the same spectral results can be proved demanding neither the global simplicity of \( \lambda_0 \) nor the symmetry with respect to \( \xi \mapsto -\xi \) of the symbol of \( H^0 \). In other words, we no longer demand the existence of a localized Wannier basis for the range of \( \pi_0 \).

2.2. Main statements. Without any loss of generality we may assume that \( \inf(\sigma(H^0)) = 0 \). Let us state our only additional hypothesis concerning the bottom of the spectrum of the unperturbed operator \( H^0 \).

**Hypothesis 2.1.** The map \( \lambda_0: T_* \rightarrow \mathbb{R} \) has a unique non-degenerate global minimum value realized for \( \theta_0 \in T_* \) and \( \lambda_0(\theta_0) = 0 \).

When \( A^\Gamma = 0 \) the above hypothesis is always satisfied [19, 4] with \( \theta_0 = 0 \) and moreover, \( \lambda_0(\theta) = \lambda_0(-\theta) \). Also, under this hypothesis and if \( b \) is small enough, the set
\[
\Sigma_b := \lambda_0^{-1}([0, b)) \subset T_*
\]
becomes a simply connected neighbourhood of the minimum.
Before we can state the main result of our paper we need some more notation.

- Given a vector potential $A$ with components of class $C^\infty_{pol}(\mathbb{R})$ and a symbol $F(x, \xi)$ we can define the following Weyl-magnetic pseudodifferential operator:

$$\text{Op}^A(F)u(x) := (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(|x-y| - \frac{1}{2} y + \frac{x}{2} + \xi \cdot y)} F\left(\frac{x+y}{2}, \frac{y}{2}\right) u(y) dy d\xi.$$  \hfill (2.4)

The magnetic pseudodifferential calculus was developed in [22]; for the convenience of the reader we summarized its main features in Section 5.

- $d_H(M_1, M_2)$ denotes the Hausdorff distance between the subsets $M_1$ and $M_2$ in $\mathbb{R}$.

**Theorem 2.2.** Let $H^{\epsilon, \kappa}$, with $(\epsilon, \kappa) \in [0,1] \times [0,1]$, be the family introduced in (1.10) for a Hamiltonian $H^0$ satisfying Hypothesis 2.1 and a magnetic field of the form (1.5). Then there exists $b > 0$ and a smooth real function $\hat{\chi}^\epsilon: \mathbb{T}_* \rightarrow \mathbb{R}$ such that

(i) for any $N \in \mathbb{N}^*$ there exist some constant $C_0 > 0$ and some $(\epsilon_0, \kappa_0) \in (0, b/N) \times (0, 1)$, such that, for any $(\epsilon, \kappa) \in (0, \epsilon_0] \times (0, \kappa_0]$,

$$d_H(\sigma(H^{\epsilon, \kappa}) \cap [0, N\epsilon], \sigma(\text{Op}^{\epsilon, \kappa}(\hat{\chi}^\epsilon)) \cap [0, N\epsilon]) \leq C_0(\kappa \epsilon + \epsilon^2),$$ \hfill (2.5)

where $\hat{\chi}^\epsilon: \mathbb{R} \rightarrow \mathbb{R}$ is the periodic extension of $\chi^\epsilon: \mathbb{T}_* \rightarrow \mathbb{R}$ considered as a function on $\mathbb{R}$ constant along the directions in $\mathbb{R}$, and $\text{Op}^{\epsilon, \kappa}(\hat{\chi}^\epsilon) \equiv \text{Op}^A(\hat{\chi}^\epsilon)$ is the magnetic quantization of $\hat{\chi}^\epsilon$ as in (2.4);

(ii) for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that for all $\theta \in \Sigma_b$ and $|\alpha| \leq m$ we have

$$|\partial^\alpha \hat{\chi}^\epsilon(\theta) - \partial^\alpha \lambda_0(\theta)| \leq C_m \epsilon.$$

Combining the above theorem with the results from [4] regarding the existence of spectral gaps for magnetic pseudo-differential operators like $\text{Op}^{\epsilon, \kappa}(\hat{\chi}^\epsilon)$, we will prove in Subsection 8.5 the following two corollaries.
Corollary 2.3. Assume $B^\Gamma = 0$. Then for any integer $N \geq 1$, there exist some constants $C_0, C_1, C_2 > 0$, and $(\epsilon_0, \kappa_0) \in (0, \tilde{b}/N) \times (0, 1)$, such that, for any $(\epsilon, \kappa) \in (0, \epsilon_0] \times (0, \kappa_0]$, there exist $a_0 < b_0 < a_1 < \cdots < a_N < b_N$, with $a_0 = \inf\{\sigma(H^{\epsilon, \kappa})\}$ so that

$$\sigma(H^{\epsilon, \kappa}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k], \quad \dim(\text{Ran } E_{[a_k, b_k]}(H^{\epsilon, \kappa})) = +\infty,$$

$$b_k - a_k \leq C_0 \epsilon (\kappa + C_1 \epsilon^{1/3}), \quad \text{for } 0 \leq k \leq N,$$

$$a_{k+1} - b_k \geq \frac{1}{C_2} \epsilon, \quad \text{for } 0 \leq k \leq N - 1.$$

Corollary 2.4. Assume $B^\Gamma \neq 0$. Under Hypothesis 2.1, for any integer $N \geq 1$, there exist positive constants $C_0, C_1, C_2$, and $(\epsilon_0, \kappa_0) \in (0, \tilde{b}/N) \times (0, 1)$, such that for any $(\epsilon, \kappa) \in (0, \epsilon_0] \times (0, \kappa_0]$ there exist $a_0 < b_0 < a_1 < \cdots < a_N < b_N$, with $a_0 = \inf\{\sigma(H^{\epsilon, \kappa})\}$ so that

$$\sigma(H^{\epsilon, \kappa}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k], \quad \dim(\text{Ran } E_{[a_k, b_k]}(H^{\epsilon, \kappa})) = +\infty,$$

$$b_k - a_k \leq C_0 \epsilon (\kappa + C_1 \epsilon^{1/5}) \quad \text{for } 0 \leq k \leq N,$$

$$a_{k+1} - b_k \geq \frac{1}{C_2} \epsilon, \quad \text{for } 0 \leq k \leq N - 1.$$

As a final remark, let us notice that in the case when the global minimum of the band $\lambda_0$ is attained at several non degenerate points, one can repeat the argument given in Section 4 of [4] making cut-offs around each minimum and the results remain true.

2.3. A roadmap of the proof. (1) We start by constructing a global smooth section $\tilde{\psi}_0 : T_+ \to \mathcal{F}$ of norm 1 vectors in $\mathcal{F}_\theta$ that coincides with the Bloch eigenvector $\tilde{\phi}_0(\theta)$ on the neighborhood $\Sigma_b$ of the global minimum $\theta_0$ of $\lambda_0$. Then the range of its associated orthogonal projection $\pi := \cup_{\Gamma}^{-1} \tilde{\pi} \cup_{\Gamma}$ in $\mathcal{H}$ will have an orthonormal basis of localized functions given by all the $\Gamma$-translations of $\tilde{\psi}_0 := \cup_{\Gamma}^{-1} \tilde{\phi}_0$. This will be presented in Section 3. An important observation is that although the projection $\pi$ does not commute with $H^0$, it still contains information about the lowest part of the spectrum of $H^0$. This is because the bounded operator $\pi H^0 \pi$ has exactly one Bloch band $\tilde{\lambda}_0(\theta)$ which coincides with $\lambda_0(\theta)$ on $\Sigma_b$ (a fact proved in Proposition 8.13).
We notice that the operator $\pi \perp H^0 \pi \perp$ seen as acting in $\pi \perp \mathcal{H}$ is bounded from below by $b$. Up to a use of the Feshbach–Schur argument (see for example [13]) and keeping the spectral parameter $E$ in the interval $[0, E_0]$ with $E_0 < b/2$, the operator $H^0 - E$ is invertible if and only if the reduced operator

$$S(E) := \pi (H^0 - E) \pi - \pi H^0 \pi \perp [\pi \perp (H^0 - E) \pi \perp]^{-1} \pi \perp H^0 \pi$$

is invertible in $\pi \mathcal{H}$.

Since we are only interested in values of $E$ which are close to zero, we may expand the reduced resolvent around $E = 0$ and obtain

$$S(E) = \pi H^0 \pi - \pi H^0 \pi \perp [\pi \perp H^0 \pi \perp]^{-1} \pi \perp H^0 \pi - E (\pi + \pi H^0 \pi \perp [\pi \perp H^0 \pi \perp]^{-2} \pi \perp H^0 \pi) + O(E^2_0).$$

We observe that the operator coefficient of $E$, i.e.

$$Y := \pi + \pi H^0 \pi \perp [\pi \perp H^0 \pi \perp]^{-2} \pi \perp H^0 \pi$$

is bounded and satisfies $Y \geq \pi$. Thus $S(E)$ is invertible in $\pi \mathcal{H}$ if and only if $Y^{-1/2} S(E) Y^{-1/2}$ is invertible in $\pi \mathcal{H}$, and some elementary arguments (see the proof of Proposition 4.2) allow to obtain an estimation of the Hausdorff distance between the spectrum of the “dressed” operator

$$\tilde{H} := Y^{-1/2} \{ \pi H^0 \pi - \pi H^0 \pi \perp [\pi \perp H^0 \pi \perp]^{-1} \pi \perp H^0 \pi \} Y^{-1/2}$$

restricted to the interval $[0, E_0]$ and the spectrum of $H^0$ in the same interval.

We notice that due to our construction of a special Wannier-like basis for the range of $\pi$, the “dressed” operator $\tilde{H}$ can be identified with a matrix acting in $\ell^2(\Gamma)$. The details will be presented in Section 4 as a consequence of a more abstract theorem.

The crucial step (explained below) is to extend these objects to the magnetic case and construct a magnetic matrix acting in $\ell^2(\Gamma)$, whose spectrum has not only gaps, but it also lies close enough to the spectrum of the full operator $H^{\epsilon,\kappa}$ near zero such that some gaps must also appear in the spectrum of $H^{\epsilon,\kappa}$. In Section 6 we use the procedure developed in [25, 14, 5, 6] in order to define a magnetic quasi-band projection $\pi^{\epsilon,\kappa}$ which is the magnetic version of $\pi$ and its range is again spanned by some localized functions which are indexed by $\Gamma$. An important observation is that $(\pi^{\epsilon,\kappa})^\perp H^{\epsilon,\kappa} (\pi^{\epsilon,\kappa})^\perp$ is still bounded from below by $b/2$ if $\epsilon$ is small enough.
(4) We can repeat the above Feshbach–Schur argument for the pair \((H^{\epsilon, \kappa}, \pi^{\epsilon, \kappa})\) with \(E_0 = N\epsilon\) where \(N\) is a fixed large number and construct a dressed magnetic matrix \(\tilde{H}^{\epsilon, \kappa}\) acting in \(\ell^2(\Gamma)\), such that its spectrum in the interval \([0, N\epsilon]\) is at a Hausdorff distance of order \(\epsilon^2\) from the spectrum of the full operator \(H^{\epsilon, \kappa}\). Hence if we can prove that the matrix \(\tilde{H}^{\epsilon, \kappa}\) has gaps or order \(\epsilon\) in the interval \([0, N\epsilon]\), the same must be true for \(H^{\epsilon, \kappa}\).

(5) In Section 8 we construct a “quasi-band” periodic and smooth function \(\tilde{\kappa}^\epsilon(\theta)\) which is close to \(\lambda_0\) on \(\Sigma_b\) such that the Hausdorff distance between the spectrum of \(\tilde{H}^{\epsilon, \kappa}\) (acting on \(\ell^2(\Gamma)\)) and the spectrum of the magnetic quantization \(\Delta p^{\epsilon, \kappa}(\tilde{\kappa}^\epsilon)\) (acting on \(L^2(\mathcal{X})\)) is of order \(\epsilon\kappa\). This leads to (2.5).

3. The quasi Wannier system

Let us spell out a straightforward but important consequence of Hypothesis 2.1.

**Lemma 3.1.** There exists \(\tilde{b} > 0\) such that, for every \(0 < b \leq \tilde{b}\),

- the set \(\Sigma_b\) is diffeomorphic to the open unit disc in \(\mathbb{R}^2\), has a smooth boundary and contains \(\theta_0\);
- the function \(\lambda_0\) is smooth on \(\Sigma_b\) and its Hessian matrix is positive;
- for \(\theta\) outside of \(\Sigma_b\) we have \(\tilde{H}^0(\theta) \geq b\).

Let us fix some \(b \in (0, \tilde{b})\) and consider the local smooth section

\[
\tilde{\phi}_0|_{\Sigma_b} : \Sigma_b \longrightarrow \mathcal{F}|_{\Sigma_b}.
\]

**Proposition 3.2.** There exists a global smooth section of norm 1 vectors

\[
\hat{\psi}_0 : \Gamma^* \longrightarrow \mathcal{F}, \quad (3.1)
\]

such that \(\hat{\psi}_0(\theta) = \tilde{\phi}_0(\theta)\) for any \(\theta \in \Sigma_b\).

**Proof.** Let \(f_j \in C^\infty_0(\mathcal{X})\), \(j \in \{1, 2\}\), with unit norms in \(L^2(\mathcal{X})\) such that both of them have support in \(|x| \leq 1/10\) and moreover, \(f_1(x) f_2(x) = 0\) for all \(x\). Let us define

\[
\hat{f}_j(x, \theta) = \sum_{\gamma \in \Gamma} e^{i\langle \theta, \gamma \rangle} f_j(x - \gamma).
\]
These functions are smooth in $x$ and belong to $\mathcal{F}_\theta$. We can check that
\[ \hat{f}_1(x, \theta) \hat{f}_2(x, \theta) = 0 \quad \text{for all } x \in E, \]
thus their scalar product in $\mathcal{F}_\theta$ equals zero. Moreover, both have norm one at fixed $\theta$.

When $\theta$ is close to $\theta_0$, the eigenvector $\hat{\phi}(x, \theta)$ can be chosen to be smooth as a function of $x$ due to elliptic regularity. It is also smooth as a function of $\theta$ on a small neighborhood of $\theta_0$. Now the vector $\hat{\phi}(\cdot, \theta_0)$ is certainly not parallel with both $\hat{f}_j(\cdot, \theta_0)$ and there must exist a $j$ (assume without loss of generality that $j = 1$) such that
\[ |\langle \hat{\phi}(\theta_0), \hat{f}_1(\cdot, \theta_0) \rangle| \leq 1/\sqrt{2}. \]
Due to continuity in $\theta$, there exists a small ball $B_r(\theta_0) \subset \bar{\Sigma}_b$ where
\[ |\langle \hat{\phi}(\theta), \hat{f}_1(\cdot, \theta) \rangle| \leq 3/4, \quad \text{for all } \theta \in B_r(\theta_0). \]

Let $0 \leq g(\theta) \leq 1$ with support in $B_r(\theta_0)$, equal to one on $B_{r/2}(\theta_0)$. Define
\[ \hat{\psi}_0(x, \theta) := \hat{h}(x, \theta) / \| \hat{h}(\cdot, \theta) \| \in \mathcal{F}_\theta \]
Then $\| \hat{\psi}_0(x, \theta) \| \leq 1/8$ hence $\hat{\psi}_0(x, \theta) := \hat{h}(x, \theta) / \| \hat{h}(\cdot, \theta) \| \in \mathcal{F}_\theta$ is a smooth extension of $\hat{\phi}$ in both $x$ and $\theta$.

We are now ready to define the principal *quasi Wannier function* $\psi_0$ by the formula
\[ \psi_0 := \mathcal{F}^{-1}_\Gamma \hat{\psi}_0 \in L^2(\mathcal{X}), \quad (3.2) \]
where $\mathcal{F}_\Gamma$ has been introduced in (1.13). The above function has norm 1 in $\mathcal{H}$, it is smooth on $\mathcal{X}$ and has fast decay together with all its derivatives. Also, the corresponding family
\[ \psi_\gamma := \tau_\gamma \psi_0 \in L^2(\mathcal{X}), \quad \gamma \in \Gamma, \quad (3.3) \]
forms an orthonormal system in $L^2(\mathcal{X})$ that we shall call the *quasi Wannier system associated with the energy window* $[0, b]$. We shall also denote by $\pi \in \mathcal{L}(\mathcal{H})$ the orthogonal projection on the closed linear subspace generated by the quasi Wannier system
\[ \mathcal{H}_0 := \text{Span}\{ \psi_\gamma : \gamma \in \Gamma \} \subset L^2(\mathcal{X}). \quad (3.4) \]
We shall use the notation
\[ \hat{\pi}(\theta) := |\hat{\psi}_0(\theta)\rangle \langle \hat{\psi}_0(\theta)|. \quad (3.5) \]
The following statement can be proven by the same arguments as Proposition 3.12 of [4].
Proposition 3.3. The vector $\psi_0$ defined in (3.2) belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Moreover, if we consider the Weyl symbol $p_0$ in $\mathcal{S}(\mathbb{R})$ of the orthogonal projection $|\psi_0\rangle\langle\psi_0|$

$$p_0(x, \xi) = (2\pi)^{-1} \int e^{i\langle \xi, v \rangle} \psi_0(x + (v/2))\overline{\psi_0(x - (v/2))} dv,$$

then the series $\sum_{\gamma \in \Gamma} p_0(x - \gamma, \xi)$ converges pointwise with all its derivatives to a $\Gamma$-periodic symbol $p \in S^{-\infty}(\mathbb{R})$ and we have

$$\pi = \mathcal{D}p(p), \quad \mathcal{U}_\Gamma \pi \mathcal{U}_\Gamma^{-1} = \int_{\mathbb{T}^n} |\hat{\psi}_0(\theta)\rangle\langle\hat{\psi}_0(\theta)| d\theta, \quad (3.6)$$

and if $K_p$ denotes the distribution kernel of $\pi$ then

$$K_p(x, y) := \sum_{\gamma \in \Gamma} \psi_0(x - \gamma)\overline{\psi_0(y - \gamma)}.$$

Proposition 3.4. With $H^0$ introduced in (1.4) and $\pi$ in (3.6), we have $\pi\mathcal{H} \subset \mathcal{D}(H^0)$, $H^0\pi \in \mathcal{L}(\mathcal{H})$ and $\pi H^0$ has a bounded closure.

Proof. Due to the properties of $\psi_0$ listed after its definition in (3.2) and due to the fact that $H^0$ is a differential operator with polynomially bounded coefficients, the quasi Wannier function $\psi_0$ belongs to $\mathcal{D}(H^0)$. Moreover, due to the rapid decay of the quasi Wannier function and its derivatives, it follows that $\mathcal{H}_0$ (defined in (3.4)) belongs in fact to $\mathcal{D}(H^0)$. We conclude that $\mathcal{H}_0 = \pi\mathcal{H} \subset \mathcal{D}(H^0)$ and thus $H^0\pi$ is a well defined bounded linear operator on $\mathcal{H}$. Moreover, the operator $\pi H^0 : \mathcal{D}(H^0) \to \mathcal{H}$ is a restriction of the adjoint $(H^0\pi)^*$ and thus has a bounded closure in $\mathcal{H}$. \qed

4. An abstract reduction argument

We consider a more abstract setting and give the details of the arguments sketched in the third item of Subsection 2.3 that gives a procedure to apply the Feshbah-Schur method in a more complicate situation when the commutator $[H, \pi]$ is no longer controlled by a small parameter (as was the situation in our previous paper [4]).
Hypothesis 4.1. We consider a triple \((H, \Pi, \beta)\) where \(H\) is a positive self-adjoint operator, \(\Pi\) is an orthogonal projection such that \(H\Pi\) is bounded (as well as \(\Pi H\)), \(\beta > 0\) and
\[
\Pi^\perp H \Pi^\perp \geq 2\beta \Pi^\perp. \tag{4.1}
\]

This implies that \(\Pi^\perp (H - E) \Pi^\perp\) is invertible in \(\Pi^\perp \mathcal{H}\) for \(E \in [0, 2\beta]\) and we denote by \(R_\perp(E) \in \mathcal{L}(\Pi^\perp \mathcal{H})\) its inverse. The spectral theorem gives
\[
\sup_{E \in [0,\beta]} \|R_\perp(E)\| \leq \beta^{-1}. \tag{4.2}
\]

Starting from
\[
Y := \Pi + \Pi H \Pi^\perp R_\perp(0)^2 \Pi^\perp H \Pi, \tag{4.3}
\]
and noting that
\(Y \geq \Pi\),
we introduce
\[
\tilde{H} := Y^{-1/2} [\Pi H \Pi - \Pi H \Pi^\perp R_\perp(0) \Pi^\perp H \Pi] Y^{-1/2} \in \mathcal{L}(\Pi \mathcal{H}). \tag{4.4}
\]

Proposition 4.2. For any \(\beta' < \beta\) we have
\[
d_H\{\sigma(H) \cap [0, \beta'], \sigma(\tilde{H}) \cap [0, \beta']\} \leq \|H\Pi\|^2 (\beta')^2 \beta^{-3}, \tag{4.5}
\]
where \(\tilde{H}\) is defined in (4.4).

Proof. Using the Feshbach–Schur reduction we get that if \(E \in [0, \beta]\) then the operator \(H - E\) is invertible if and only if the operator
\[
S(E) := \Pi (H - E) \Pi - \Pi H R_\perp(E) H \Pi \tag{4.6}
\]
is invertible in \(\Pi \mathcal{H}\). In this case we have the identity
\[
\Pi (H - E)^{-1} \Pi = S(E)^{-1}. \tag{4.7}
\]

Using the resolvent equation, we have, if \(E \in [0, \beta']\),
\[
R_\perp(E) = R_\perp(0) + R_\perp(0)^2 + X(E), \quad X(E) := E^2 R_\perp(0)^2 R_\perp(E),
\]
\[
\|X(E)\| \leq E^2 \beta^{-3} \leq \beta'^2 \beta^{-3}.
\]
Thus we can write $S(E)$ as

$$S(E) = \Pi H \Pi - \Pi H R \perp (0) H \Pi - E(Y - \Pi) - E \Pi + \Pi H X(E) H \Pi$$

$$= \Pi [H - HR \perp (0) H] \Pi - EY + \Pi H X(E) H \Pi.$$

Since $Y$ in (4.3) is bounded from below by 1, we can define the positive operator $Y^{-1/2}$ in $\mathcal{L}(\Pi \mathcal{H})$. Then (use the notation (4.4) in the above identity) $S(E)$ is invertible if and only if

$$Y^{-1/2} S(E) Y^{-1/2} = \tilde{H} - E + Y^{-1/2} \Pi H X(E) H \Pi Y^{-1/2}$$

is invertible in $\Pi \mathcal{H}$ with bounded inverse. We are now prepared to prove Proposition 4.2 and we do it in two steps.

**Step 1.** Assume that $E \in [0, \beta']$ is in the resolvent set of $\tilde{H}$. Then (4.8) implies

$$S(E) = Y^{1/2} \{1 + Y^{-1/2} \Pi H X(E) H \Pi Y^{-1/2} (\tilde{H} - E)^{-1}\}(\tilde{H} - E) Y^{1/2}.$$

We have

$$\|Y^{-1/2} \Pi H X(E) H \Pi Y^{-1/2} (\tilde{H} - E)^{-1}\| \leq \frac{\beta'^2 \|H \Pi\|^2 \beta^{-3}}{\text{dist}(E, \sigma(\tilde{H})), \text{ if } E \in [0, \beta']}.$$

Thus, if $\text{dist}(E, \sigma(\tilde{H})) > \beta'^2 \|H \Pi\|^2 \beta^{-3}$, $S(E)$ is invertible, hence $E$ is in the resolvent set of $H$. In other words, no element of $\sigma(H) \cap [0, \beta']$ can be situated at a distance which is larger than $\beta'^2 \|H \Pi\|^2 \beta^{-3}$ from $\sigma(\tilde{H}) \cap [0, \beta']$.

**Step 2.** Assume that $E \in [0, \beta']$ is in the resolvent set of $H$ hence (4.7) holds. From (4.8) we see that $\tilde{H} - E$ is invertible when the operator

$$S(E) - \Pi H X(E) H \Pi = \{1 - \Pi H X(E) H \Pi (H - E)^{-1} \Pi\} S(E)$$

is invertible.

As before we have

$$\|\Pi H X(E) H \Pi (H - E)^{-1} \Pi\| \leq \frac{\beta'^2 \|H \Pi\|^2 \beta^{-3}}{\text{dist}(E, \sigma(\tilde{H})), \text{ if } E \in [0, \beta')}.$$

Hence $\tilde{H} - E$ is invertible when $\text{dist}(E, \sigma(\tilde{H})) > \beta'^2 \|H \Pi\|^2 \beta^{-3}$. Equivalently, no element of $\sigma(\tilde{H}) \cap [0, \beta']$ can be situated at a distance larger than $\beta'^2 \|H \Pi\|^2 \beta^{-3}$ from $\sigma(H) \cap [0, \beta']$.

\[\square\]
Corollary 4.3. Let $0 < D_1 < D_2 < \beta' < \beta$ and assume $(D_1, D_2) \cap \sigma(\tilde{H}) = \emptyset$. Then, if

$$\|H \Pi\|^2 (\beta')^2 \beta^{-3} < \frac{1}{2} (D_2 - D_1)$$

we have

$$(D_1 + \|H \Pi\|^2 (\beta')^2 \beta^{-3}, D_2 - \|H \Pi\|^2 (\beta')^2 \beta^{-3}) \cap \sigma(H) = \emptyset.$$ 

It will be interesting to apply this corollary for a family indexed by $\eta \in [0, \epsilon_1]$ of triples $(H(\eta), \Pi(\eta), \beta)$ satisfying (4.1). This leads to

Corollary 4.4. Let $0 < D_1 < D_2 < \beta' < \beta$ and assume $(D_1, D_2) \cap \sigma(\tilde{H}(\eta)) = \emptyset$, for all $\eta \in [0, \epsilon_1]$. Then, if

$$D := \sup_{\eta \in [0, \epsilon_1]} \|H(\eta)\Pi(\eta)\|^2 (\beta')^2 \beta^{-3} < \frac{1}{2} (D_2 - D_1),$$

we have

$$(D_1 + D, D_2 - D) \cap \sigma(H(\eta)) = \emptyset.$$ 

Proof. It is a direct consequence of Corollary 4.3. \qed

Finally, the next proposition gives sufficient conditions for the appearance of gaps of order $\eta$ in the spectrum of $H(\eta)$.

Proposition 4.5. Suppose that there exist two numerical constants $0 < C_1 < C_2$ such that the interval $(C_1, C_2)$ belongs to the resolvent set of $\eta^{-1} \tilde{H}(\eta)$ for all $0 < \eta \leq \min(\epsilon_1, \frac{\beta}{C_2 + 1})$. Let

$$C_0 := (C_2 + 1)^2 (\sup_{\eta \in [0, \epsilon_1]} \|H(\eta)\Pi(\eta)\|^2 \beta^{-3}),$$

$$\epsilon_0 := \min \left( \epsilon_1, \frac{\beta}{C_2 + 1}, \frac{1}{2} \frac{C_2 - C_1}{C_0} \right).$$

Then,

$$(C_1 \eta + C_0 \eta^2, C_2 \eta - C_0 \eta^2) \cap \sigma(H(\eta)) = \emptyset, \quad \text{for all } \eta \in (0, \epsilon_0).$$

Proof. Let $D_1 = C_1 \eta$, $D_2 = C_2 \eta$, $\beta' = (C_2 + 1) \eta$. We then apply Corollary 4.4. \qed
Let us recall from [4] the following two notations:

\[ \Lambda^A(x, y) := e^{-i \int_{[x,y]} A}, \quad \Omega^B(x, y, z) := e^{-i \int_{(x,y,z)} B}, \tag{5.1} \]

where \([x, y]\) denotes the oriented interval from \(x\) to \(y\) and \((x, y, z)\) denotes the oriented triangle with vertices \(x, y, z\), with the integrals being the usual invariant integrals of \(k\)-forms for \(k = 1, 2\).

We shall also use the following shorthand notation:

\[ \Lambda^\epsilon \equiv \Lambda^{\epsilon A_0}, \quad \Lambda^{\epsilon, \kappa} \equiv \Lambda^{A^{\epsilon, \kappa}}, \quad \Omega^\epsilon \equiv \Omega^{\epsilon B_0}, \quad \Omega^{\epsilon, \kappa} \equiv \Omega^{B^{\epsilon, \kappa}}, \tag{5.2a} \]

\[ \tilde{\Lambda}^{\epsilon, \kappa} \equiv \Lambda^{k A^\epsilon}, \quad \tilde{\Omega}^{\epsilon, \kappa} \equiv \Omega^{k B \epsilon}. \tag{5.2b} \]

We notice for future use that

\[ \Lambda^A(x, z) \lambda^A(z, y) \lambda^A(y, x) = \Omega^B(x, z, y), \tag{5.3} \]

and that there exists \(C > 0\) such that, for any magnetic field \(B\) of class \(BC^\infty(X)\),

\[ |\Omega^B(x, y, z) - 1| \leq C \|B\|_\infty |(y - x) \wedge (z - x)|. \tag{5.4} \]

We notice also that for our choice of gauge for \(A_0\) we have the relations

\[ \Lambda^\epsilon(x, y) = \exp\{i (B_0/2) x \wedge y\}, \tag{5.5} \]

\[ \Omega^\epsilon(x, y, z) = \exp\{i B_0 (x - y) \wedge (z - y)\}. \tag{5.6} \]

Let us recall the gauge covariance property which states that whatever magnetic potential \(A\) we choose for a given magnetic field \(B(x) := B_{12}(x) = -B_{21}(x)\) we have

\[ \{ -i \nabla - A(x) \}^2 \lambda^A(x, z) = \lambda^A(x, z) \{ -i \nabla - a(x, z) \}^2, \tag{5.7} \]

where

\[ a_j(x, z) = \sum_k (x - z)_k \int_0^1 B_{jk}(z + s(x - z))s ds \text{ for } j = 1, 2. \tag{5.8} \]

Let \(\mathbb{X}^* \cong \mathbb{R}^2\) be the dual of our configuration space \(\mathbb{X} \cong \mathbb{R}^2\) and denote by \(\langle \cdot, \cdot \rangle\) the duality map \(\mathbb{X}^* \times \mathbb{X} \to \mathbb{R}\) between these two spaces. Let \(\Sigma := \mathbb{X} \times \mathbb{X}^*\) be the associated phase space with the canonical symplectic structure

\[ \sigma((x, \xi), (y, \eta)) := \langle \xi, y \rangle - \langle \eta, x \rangle, \quad \text{for all } ((x, \xi), (y, \eta)) \in \Sigma \times \Sigma. \tag{5.9} \]
We will use the following class of Hörmander type symbols.

**Definition 5.1.** For any $s \in \mathbb{R}$ we denote by

$$S^s(\Xi) := \{ F \in C^\infty(\Xi) \mid v_{n,m}^s(F) < +\infty, \text{ for all } (n,m) \in \mathbb{N} \times \mathbb{N} \},$$

where

$$v_{n,m}^s(f) := \sup_{(x,\xi) \in \Xi} \sum |\alpha| \leq n \sum |\beta| \leq m \langle (\xi)^{-s+m} (\partial_x^\alpha \partial_\xi^\beta f)(x,\xi) \rangle,$$

and

$$S^-(\Xi) := \bigcup_{s<0} S^s(\Xi) \quad \text{and} \quad S^{-\infty}(\Xi) := \bigcap_{s \in \mathbb{R}} S^s(\Xi). \quad (5.10)$$

**Definition 5.2.** A symbol $F$ in $S^s(\Xi)$ is called **elliptic** if there exist two positive constants $R$ and $C$ such that

$$|F(x,\xi)| \geq C \langle \xi \rangle^s,$$

for any $(x,\xi) \in \Xi$ with $|\xi| \geq R$.

**Definition 5.3.** We consider a vector potential $A$ with components of class $C^\infty_{pol}(\mathcal{X})$. For any $F \in S(\Xi)$ we can define ([22]) the linear operator

$$u \mapsto (\mathcal{O}p^A(F)u)(x) := (2\pi)^{-2} \int_{\Xi} \int_{\Xi^*} e^{i \langle x, x-y \rangle} e^{-i \int_{[x,y]} A F(\frac{x+y}{2},\xi) u(y) d\xi} dy, \quad (5.11)$$

for all $u \in S(\mathcal{X})$.

This operator is continuous on $S(\mathcal{X})$ and has a natural extension by duality to $S'(\mathcal{X})$. For $A = 0$ we obtain the usual Weyl calculus. In [22] it is proven the following result.

**Proposition 5.4.** For any vector potential $A$ with components of class $C^\infty_{pol}(\mathcal{X})$ the quantization $\mathcal{O}p^A$ defines a linear and topological isomorphism between $S'(\Xi)$ and the linear topological space of continuous operators from $S(\mathcal{X})$ to $S'(\mathcal{X})$.

We shall denote by $\mathcal{S}^A$ its inverse, i.e. the application that associates to a given operator the tempered distribution which is its symbol for the magnetic pseudodifferential calculus defined by the vector potential $A$:

$$\mathcal{S}^A \circ \mathcal{O}p^A = I. \quad (5.12)$$
In the same spirit as the Calderon–Vainilancourt theorem for classical pseudo-differential operators, Theorem 3.1 in [16] states that any symbol $F \in S^0_0(\mathcal{X})$ defines a bounded operator $\mathcal{O}p^A(F)$ in $L^2(\mathcal{X})$ with an upper bound of the operator norm given by some seminorm of $F$ as Hörmander type symbol. We denote by $\|F\|_B$ the operator norm of $\mathcal{O}p^A(F)$ in $\mathcal{L}(L^2(\mathcal{X}))$,

$$\|F\|_B := \|\mathcal{O}p^A(F)\|_{\mathcal{L}(L^2(\mathcal{X}))}.$$  

(5.13)

This norm only depends on the magnetic field $B$ and not on the choice of the vector potential (different choices being unitary equivalent).

We notice that

$$H^0 = \mathcal{O}p^A(\Gamma), \quad h(x, \xi) := \xi^2 + V(\Gamma)(x).$$  

(5.14)

and for the perturbed Hamiltonians we will use the following shorthand notations

$$H^{\epsilon, \kappa} = \mathcal{O}p^A^{\epsilon, \kappa}(h) =: \mathcal{O}p^\epsilon(h), \quad H^\epsilon = \mathcal{O}p^A(\epsilon)(h) =: \mathcal{O}p^\epsilon(h).$$  

(5.15)

We also recall from [22] that for any two test functions $f$ and $g$ in $S(\mathcal{X})$ the product of the linear operators $\mathcal{O}p^A(f)$ and $\mathcal{O}p^A(g)$ induces a twisted Moyal product, also called magnetic Moyal product, such that

$$\mathcal{O}p^A(f) \mathcal{O}p^A(g) = \mathcal{O}p^A(f \#^B g).$$

This product depends only on the magnetic field $B$ and is given by an explicit oscillating integral. The following relation proven in [22] (Lemme 4.14):

$$\int_{\mathcal{X}} d\mathcal{X}(f \#^B g)(X) = \int_{\mathcal{X}} d\mathcal{X}f(X)g(X)$$  

(5.16)

allows to extend the magnetic Moyal product to a class of tempered distributions that leave $S(\mathcal{X})$ invariant by magnetic Moyal composition. In [22] it is proven that this class forms a $*$-algebra that we call the magnetic Moyal algebra and denote by $\mathfrak{M}^B$. Moreover this class contains all the Hörmander type symbols (see [22, 16]) and the usual composition of symbols theorem is still valid (Theorem 2.2 in [16]).

For any invertible symbol $F$ in $\mathfrak{M}^B$, we denote by $F^{-1}_B$ its inverse. It is shown in Subsection 2.1 of [23] that, for any $m > 0$ and for $a > 0$ large enough (depending on $m$) the symbol $s_m(x, \xi) := (\xi)^m + a$, has an inverse in $\mathfrak{M}^B$. We shall use the shorthand notation $s_m^B$ instead of $(s_m)_B^-$ and extend it to any $m \in \mathbb{R}$ (thus for $m > 0$ we have $s_m^B \equiv s_m$).
The following results have been established in [17] (Propositions 6.2 and 6.3).

Proposition 5.5. (1) If \( F \in S^{0}(\Xi) \) is invertible in \( \mathfrak{M}^{B} \), then the inverse \( F^{-1}_{B} \) also belongs to \( S^{0}(\Xi) \).

(2) For \( m < 0 \), if \( f \in S^{m}(\Xi) \) is such that \( 1 + f \) is invertible in \( \mathfrak{M}^{B} \), then \( (1 + f)_{B}^{-1} = \mathcal{L}(L^{2}(\mathcal{X})) \), then \( G_{B}^{-1} \in S^{-m}(\Xi) \).

(3) For \( m > 0 \), if \( G \in S^{m}(\Xi) \) is invertible in \( \mathfrak{M}^{B} \), with \( \mathcal{O}p^{A}(s_{m}B G_{B}) \in \mathcal{L}(L^{2}(\mathcal{X})) \), then \( G_{B}^{-1} \in S^{-m}(\Xi) \).

From the explicit form of the magnetic Moyal product (see in [22]) we easily notice the following fact.

Proposition 5.6. Let us consider the lattice \( \Gamma_{\ast} \subset \mathcal{X}^{\ast} \). If \( f \) and \( g \) are \( \Gamma_{\ast} \)-periodic symbols, then their magnetic Moyal product is also \( \Gamma_{\ast} \)-periodic.

About the resolvent  By Theorem 4.1 in [16], for any real elliptic symbol \( h \in S^{m}(\Xi) \) (with \( m > 0 \)) and for any \( A \in C_{\text{pol}}(\mathcal{X}, \mathbb{R}^{2}) \), the operator \( \mathcal{O}p^{A}(h) \) has a closure \( H^{A} \) in \( L^{2}(\mathcal{X}) \) that is self-adjoint on a domain \( \mathcal{H}_{A}^{m} \) (a magnetic Sobolev space) and lower semibounded. Thus we can define its resolvent \( (H^{A} - \lambda)^{-1} \) for any \( \lambda \notin \sigma(H^{A}) \) and Theorem 6.5 in [17] states that it exists a symbol \( r^{B}_{\lambda}(h) \in S^{-m}(\Xi) \) such that

\[
(H^{A} - \lambda)^{-1} = \mathcal{O}p^{A}(r^{B}_{\lambda}(h)).
\]

Integral kernels and symbols  For symbols of class \( S^{0}(\Xi) \), we have seen that the associated magnetic pseudodifferential operator is bounded in \( \mathcal{H} \) and is self-adjoint if and only if its symbol is real. In that case we can also define its resolvent and the results in [17], cited above, show that it is also defined by a symbol of class \( S^{0}(\Xi) \).

Given any tempered distribution \( T \in S'(\mathcal{X} \times \mathcal{X}) \) we shall denote by

\[
\mathcal{I}nt(T) : S(\mathcal{X}) \rightarrow S'(\mathcal{X})
\]

the integral operator having the distribution kernel \( T \), given by the formula

\[
\langle u, (\mathcal{I}nt(T))v \rangle_{\mathcal{H}} := \langle T, \tilde{u} \otimes v \rangle, \quad \text{for all } (u, v) \in S(\mathcal{X}) \times S(\mathcal{X}). \tag{5.17}
\]

For any tempered distribution \( F \in S'(\Xi) \) we denote by \( K_{F} \in S'(\mathcal{X} \times \mathcal{X}) \) the integral kernel of its Weyl quantization, i.e. \( \mathcal{O}p(F) = \mathcal{I}nt(K_{F}) \). Let us recall that there exists a linear bijection \( \mathfrak{M} : S'(\Xi) \rightarrow S'(\mathcal{X} \times \mathcal{X}) \) defined by

\[
(\mathfrak{M}F)(x, y) := (2\pi)^{-2} \int_{\mathcal{X}^{\ast}} e^{i\langle \xi, x-y \rangle} F\left(\frac{x+y}{2}, \xi\right) d\xi, \tag{5.18}
\]
Peierls’ substitution for low lying spectral energy windows such that

\[ \mathcal{D}p(F) = \text{Int}(\mathcal{M}F). \]

In the magnetic calculus, we have the equality

\[ \mathcal{D}p^A(F) = \text{Int}(\Lambda^A2\mathcal{M}F). \] (5.19)

6. The magnetic quasi-band projections

Following step by step the arguments in Section 3 of [4] we can associate with the quasi-band orthogonal projection \( \pi \in \mathcal{L}(\mathcal{H}) \) some magnetic versions of it: \( \pi^\epsilon \) and \( \pi^{\epsilon,\kappa} \). Starting from the principal quasi Wannier function \( \psi_0 \) introduced in Definition 3.2 and the magnetic field introduced in (1.5) we define the following objects.

**Definition 6.1.** We set

1. \( \hat{\phi}^\epsilon_\gamma(x) := \Lambda^\epsilon(x, \gamma)\psi_0(x - \gamma), \)
2. \( \mathcal{G}_{\alpha\beta}^\epsilon := \langle \hat{\phi}_\alpha^\epsilon, \hat{\phi}_\beta^\epsilon \rangle_{\mathcal{H}}, \)
3. \( \mathcal{F}^\epsilon := (\mathcal{G}^\epsilon)^{-1/2} \in \mathcal{L}(\ell^2(\Gamma)); \)
4. \( \psi_0^\epsilon(x) := \sum_{\alpha \in \Gamma} \mathcal{F}_{\alpha0}^\epsilon \Omega^\epsilon(\alpha, 0, x)\psi_0(x - \alpha), \)
5. \( \phi^\epsilon_\gamma(x) := \Lambda^\epsilon(x, \gamma)\psi_0^\epsilon(x - \gamma); \)
6. \( \pi^\epsilon := \sum_{\gamma \in \Gamma} |\phi^\epsilon_\gamma \rangle \langle \phi^\epsilon_\gamma | = (\pi^\epsilon)^2 = (\pi^\epsilon)^*; \)
7. \( \phi^\epsilon_{\gamma,\kappa}(x) := \Lambda^{\epsilon,\kappa}(x, \gamma)\psi_0^\epsilon(x - \gamma), \)
8. \( \mathcal{G}_{\alpha\beta}^{\epsilon,\kappa} := (\hat{\phi}_\alpha^\epsilon, \hat{\phi}_\beta^\epsilon)_{\mathcal{H}}, \)
9. \( \mathcal{F}^{\epsilon,\kappa} := (\mathcal{G}^{\epsilon,\kappa})^{-1/2} \in \mathcal{L}(\ell^2(\Gamma)); \)
10. \( \phi^{\epsilon,\kappa}_{\gamma}(x) := \sum_{\alpha \in \Gamma} \mathcal{F}_{\alpha\gamma}^{\epsilon,\kappa} \phi^\epsilon_\alpha, \)
11. \( \pi^{\epsilon,\kappa} := \sum_{\gamma \in \Gamma} |\phi^{\epsilon,\kappa}_\gamma \rangle \langle \phi^{\epsilon,\kappa}_\gamma | = (\pi^{\epsilon,\kappa})^2 = (\pi^{\epsilon,\kappa})^*. \)

Using the notation introduced in (1.8)-(5.2) we notice that

\[ \Lambda^{\epsilon,\kappa}(x, y) = \tilde{\Lambda}^{\epsilon,\kappa}(x, y)\Lambda^\epsilon(x, y) \] (6.1)

so that

\[ \hat{\phi}^{\epsilon,\kappa}_\gamma(x) = \tilde{\Lambda}^{\epsilon,\kappa}(x, \gamma)\phi^\epsilon_\gamma(x). \] (6.2)
If we simply replace the Wannier function $\phi_0$ in Section 3 of [4] with the quasi Wannier function $\psi_0$, all the results of Section 3 of [4] remain true due to the fact that the only properties of $\phi_0$ that are used are its smoothness and rapid decay together with all its derivatives, and these facts are true for the quasi Wannier function too. Thus we obtain the following statement, extending the results in Section 3 of [4]:

**Proposition 6.2.** There exists $\epsilon_0 > 0$ such that, for any $\epsilon \in [0, \epsilon_0]$,  
(1) $F^\epsilon$ belongs to $\mathcal{L}(\ell^2(\Gamma)) \cap \mathcal{L}(\ell^\infty(\Gamma))$ and for any $m \in \mathbb{N}$ there exists $C_m$ such that  
$$\langle \alpha - \beta \rangle^m |F^\epsilon_{\alpha \beta} - \delta_{\alpha \beta}| \leq C_m \epsilon \quad \text{for all } (\alpha, \beta) \in \Gamma^2,$$
and there exists a rapidly decaying function $F^\epsilon : \Gamma \mapsto \mathbb{C}$ such that  
$$F^\epsilon_{\alpha \beta} = \Lambda^\epsilon (\alpha, \beta) F^\epsilon (\alpha - \beta);$$
(2) for any $m \in \mathbb{N}$ and any $a \in \mathbb{N}^2$, there exists $C_{m,a} > 0$ such that  
$$\sup_{x \in \mathcal{X}} \langle x \rangle^m |(\partial^a \psi^\epsilon_0)(x) - (\partial^a \psi_0)(x)| \leq C_{m,a} \epsilon;$$
(3) for any $m \in \mathbb{N}$, there exists $C_m > 0$ such that  
$$\sup_{(\alpha, \beta) \in \Gamma^2} \langle \alpha - \beta \rangle^m |F^{\epsilon,\kappa}_{\alpha \beta} - \delta_{\alpha \beta}| \leq C_m \kappa \epsilon, \quad \text{for all } (\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1]. \quad (6.3)$$

Replacing $\psi_0$ with $\psi^\epsilon_0$ in the proof of Proposition 3.4 we obtain the following result.

**Proposition 6.3.** There exists $\epsilon_0 > 0$ such that for any $(\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1]$ we have  
$$\pi^{\epsilon,\kappa} \mathcal{H} \subset \mathcal{D}(H^{\epsilon,\kappa}),$$
while $H^{\epsilon,\kappa} \pi^{\epsilon,\kappa} \in \mathcal{L}(\mathcal{H})$ and $\pi^{\epsilon,\kappa} H^{\epsilon,\kappa}$ has a bounded closure.

We can also construct the $\Gamma$-periodic symbol $p_\epsilon \in S^{-\infty}(\mathbb{Z})$ such that $\pi^\epsilon = \Omega p^\epsilon (p_\epsilon)$ and the symbol $p_{\epsilon,\kappa} \in S^{-\infty}(\mathbb{Z})$ such that $\pi^{\epsilon,\kappa} = \Omega p^{\epsilon,\kappa} (p_{\epsilon,\kappa})$ and the same proof as in [4] gives the following result.

**Proposition 6.4.** There exists $\epsilon_0 > 0$ such that for any seminorm $v$ on $S^{-\infty}(\mathbb{Z})$, there exists $C_v > 0$ such that  
$$v(p^\epsilon - p) \leq C_v \epsilon \quad \text{and} \quad v(p^{\epsilon,\kappa} - p^\epsilon) \leq C_v \kappa \epsilon, \quad \text{for all } (\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1].$$

Note that in this case the commutator $[H^{\epsilon,\kappa}, \pi^{\epsilon,\kappa}]$ is no longer small (due to the arbitrary deformation which was made in constructing the quasi Wannier function).
7. Checking the conditions of the abstract reduction procedure for the magnetic operators

We want to apply Propositions 4.2 and 4.5 for magnetic pseudodifferential operators and to control the behavior of their symbols. The following abstract result concerns the symbol of the “reduced resolvent” $R_{\perp}(E)$ which is introduced after (4.1). In our applications below we shall replace $H$ with $H - E\mathbb{1}$ for some $E < \inf\sigma(H)$.

Suppose given some real elliptic symbol $h \in S^m_1(\Xi)_{\text{ell}}$ for some $m > 0$ and consider the magnetic pseudodifferential calculus, denoted by $\mathcal{Op}^A$, associated with a smooth polynomially bounded vector potential $A$ such the magnetic field $B = \text{curl} A$ is of class $BC^\infty(\Xi)$. Let

$$H = \mathcal{Op}^A(h)$$

and suppose also given an orthogonal projection $\Pi = \mathcal{Op}^A(p)$ with $p \in S^{-\infty}(\Xi)$ and such that $\Pi\mathcal{H} \subset \mathcal{D}(H)$. Let us also introduce

$$H^\perp := \Pi^\perp H \Pi^\perp \quad \text{with} \quad \Pi^\perp := \mathbb{1} - \Pi.$$

**Proposition 7.1.** With the above notation, if $H^\perp$ is invertible on $\Pi^\perp\mathcal{H}$ with bounded inverse $R \in \mathcal{L}(\Pi^\perp\mathcal{H})$, then there exists $r \in S^{-m}(\Xi)$ such that $\Pi^\perp R\Pi^\perp = \mathcal{Op}^A(r)$.

**Proof.** We notice that

$$\Pi^\perp R\Pi^\perp = \Pi^\perp (\Pi + \Pi^\perp H \Pi^\perp)^{-1} \Pi^\perp$$

and use point (3) of Proposition 5.5.}

Let us verify that the pair of operators $(H^\epsilon,\pi^\epsilon)$ satisfies Hypothesis 4.1.

**Proposition 7.2.** There exist $\epsilon_0 > 0$ and $b_0 \in (0, \tilde{b})$ (with $\tilde{b} > 0$ introduced in Lemma 3.1) such that for any $(\epsilon,\kappa) \in [0,\epsilon_0] \times [0,1]$ the pair of operators $(H^\epsilon,\pi^\epsilon)$ verify the following properties:

1. $H^\epsilon,\pi^\epsilon$ is bounded on $\mathcal{H}$, uniformly in $\epsilon$ and $\kappa$;

2. $(\mathbb{1} - \pi^\epsilon)H^\epsilon(\mathbb{1} - \pi^\epsilon) \geq b_0(\mathbb{1} - \pi^\epsilon)$.

**Proof.** The first statement is a direct consequence of Proposition 6.3 and we focus on the second statement.
Let $b' < b < \tilde{b}$ with $\tilde{b} > 0$ introduced in Lemma 3.1 so that the closure of $\Sigma_{b'}$ is included in $\Sigma_b$. We choose a function $g \in C_0^\infty(\Sigma_b)$ such that $0 \leq g(\theta) \leq 1$ and $g(\theta) = 1$ on $\Sigma_{b'}$. Then we define successively (using (3.5)):

$$\hat{K}^0(\theta) := \hat{H}^0(\theta) + g(\theta)\hat{\pi}(\theta), \quad K^0 := \bigcup_{\Gamma}^{-1}\left(\int_{E^*} \hat{K}^0(\theta)d\theta\right)\bigcup_{\Gamma}, \quad (7.2)$$

$$W^0 := K^0 - H^0, \quad w := \mathbb{G}(W^0), \quad W_{\epsilon,\kappa} := \mathcal{D}p_{\epsilon,\kappa}(w), \quad K_{\epsilon,\kappa} := H_{\epsilon,\kappa} + W_{\epsilon,\kappa}. \quad (7.3)$$

By construction, we have $K^0 \geq b' \mathbb{1}$, which implies

$$\pi^\perp H^0 \pi^\perp = \pi^\perp K^0 \pi^\perp \geq b' \pi^\perp. \quad (7.4)$$

Using the regularity of the spectrum (see Corollary 1.6 in [8]), we conclude that for $\epsilon_0 > 0$ small enough, $K_{\epsilon,\kappa} \geq (3/4)b'$ for every $\epsilon \in [0, \epsilon_0]$ and $0 \leq \kappa \leq 1$, and this implies

$$(\mathbb{1} - \pi_{\epsilon,\kappa})K_{\epsilon,\kappa}(\mathbb{1} - \pi_{\epsilon,\kappa}) \geq (3/4)b'(\mathbb{1} - \pi_{\epsilon,\kappa}). \quad (7.5)$$

By construction, we have that $W^0 = \pi W^0 \pi$. We show that their magnetic counterparts are also close in norm. We write

$$\pi_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa} - W_{\epsilon,\kappa} = \mathcal{D}p_{\epsilon,\kappa}\left(p_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa} - W_{\epsilon,\kappa} - p_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa}\right),$$

while

$$p_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa} - W_{\epsilon,\kappa} = \mathcal{D}p_{\epsilon,\kappa}\left(p_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa} - p_{\epsilon,\kappa}w\pi p\right)$$

$$= (p_{\epsilon,\kappa}W_{\epsilon,\kappa} - p_{\epsilon,\kappa}w\pi p) + p_{\epsilon,\kappa}(w\pi p - w\pi p) + (p_{\epsilon,\kappa} - p)\pi w\pi p$$

is a symbol of class $S^0_1(\Xi) \subset \mathcal{C}^{\epsilon,\kappa}(\Xi)$ having the norm in $\mathcal{C}^{\epsilon,\kappa}(\Xi)$ (i.e. the operator norm of its magnetic quantization $\mathcal{D}p_{\epsilon,\kappa}$) of order $\epsilon$ and thus there exist $C$ and $\epsilon_0$ such that

$$\|\pi_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa} - W_{\epsilon,\kappa}\| \leq C \epsilon, \quad (7.6)$$

for any $(\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1]$.

Finally, writing

$$(\mathbb{1} - \pi_{\epsilon,\kappa})H_{\epsilon,\kappa}(\mathbb{1} - \pi_{\epsilon,\kappa}) = (\mathbb{1} - \pi_{\epsilon,\kappa})K_{\epsilon,\kappa}(\mathbb{1} - \pi_{\epsilon,\kappa})$$

$$- (\mathbb{1} - \pi_{\epsilon,\kappa})(W_{\epsilon,\kappa} - \pi_{\epsilon,\kappa}W_{\epsilon,\kappa}\pi_{\epsilon,\kappa})(\mathbb{1} - \pi_{\epsilon,\kappa})$$

and using (7.5) and (7.6) we see that if $\epsilon$ is small enough then we can choose $b_0 = b'/2$. This achieves the proof. \qed
Thus Hypothesis 4.1 is satisfied when $H$ and $\Pi$ are replaced by $H^{\epsilon,k}$ and $\pi^{\epsilon,k}$ respectively, and $\beta = b'/2$. Hence Proposition 4.5 can be applied and fixing $N > 1$ and working with $\epsilon \geq 0$ such that $\epsilon N < b'/2$, we deduce that $H^{\epsilon,k}$ has $N$ gaps of order $\epsilon$ in its lower spectrum if the operator $\tilde{H}^{\epsilon,k}$ associated with the pair $(H^{\epsilon,k}, \pi^{\epsilon,k})$ as in (4.3) and (4.4) also has $N$ gaps of order $\epsilon$ in its lower spectrum. We shall use the notations $\gamma^{\epsilon,k} \in \mathcal{L}(\mathcal{H})$ and $\tilde{H}^{\epsilon,k} \in \mathcal{L}(\pi^{\epsilon,k}\mathcal{J})$ for the operators defined as in (4.3) and (4.4) for $H$ replaced by $H^{\epsilon,k}$ and $\Pi$ replaced by $\pi^{\epsilon,k}$.

8. The One Band Effective Hamiltonian

The conclusion of Section 7 implies that in order to finish the proof of our main result in Theorem 2.2 we have to analyze the bottom of the spectrum of the operator $\pi^{\epsilon,k} \tilde{H}^{\epsilon,k} \pi^{\epsilon,k}$. We follow essentially the arguments and ideas from Subsection 3.3 in [4], but due to the fact that in the present situation the calculus is much more involved, we prefer to present rather complete proofs.

8.1. Preliminaries. As we shall start with the situation with constant magnetic field, let us recall the following results concerning the Zak magnetic translations [31] (see also [25, 14, 5]).

**Proposition 8.1.** In the case of a constant magnetic field of the form $B_\epsilon = \epsilon B_0$ the family of unitary operators

$$\mathcal{T}_\gamma^{\epsilon} := \Lambda^{\epsilon}(\cdot, \gamma)\tau_{\gamma}, \quad \gamma \in \Gamma$$

satisfies the following properties:

1. $\mathcal{T}_{\alpha}^{\epsilon} \mathcal{T}_{\beta}^{\epsilon} = \Lambda^{\epsilon}(\beta, \alpha)\mathcal{T}_{\alpha+\beta}^{\epsilon}$;
2. the operator $\mathcal{Dp}^{\epsilon}(F)$ commutes with all the $\{\mathcal{T}_\gamma^{\epsilon}\}_{\gamma \in \Gamma}$ if and only if $F \in S'(\Xi)$ is $\Gamma$-periodic with respect to the variable in $\Xi$.

On the other hand, looking at the definition of the magnetic Moyal product and using the second point of the above proposition one can prove the following statement.

**Proposition 8.2.** For a constant magnetic field $B_\epsilon := \epsilon B_0$ the following statements hold true:

1. the subspace $\mathcal{M}_\Gamma^{B_\epsilon}$ of the $\Gamma$-periodic distributions of $\mathcal{M}^{B_\epsilon}$ is a subalgebra;
2. if $F \in \mathcal{M}_\Gamma^{B_\epsilon}$ is invertible in $\mathcal{M}^{B_\epsilon}$, its inverse is in $\mathcal{M}_\Gamma^{B_\epsilon}$. 

Having in mind Proposition 5.4 and with the shorthand notation (see (5.12))

\[ \mathcal{S}^{\epsilon,k} := \mathcal{S}A^{\epsilon,k}, \quad \mathcal{S}^{\epsilon} := \mathcal{S}A^{\epsilon}, \]  

(8.2)

we introduce the following symbols.

**Definition 8.3.** We set

1. \( h^{\epsilon,k}_0 := \mathcal{S}^{\epsilon,k}(\pi^{\epsilon,k} H^{\epsilon,k} \pi^{\epsilon,k}) \),
2. \( h^{\epsilon}_0 := \mathcal{S}^{\epsilon}(\pi^{\epsilon} H^{\epsilon} \pi^{\epsilon}) \);
3. \( h^{\epsilon,k}_\perp := \mathcal{S}^{\epsilon,k}((\mathbf{1} - \pi^{\epsilon,k}) H^{\epsilon,k}(\mathbf{1} - \pi^{\epsilon,k})) \),
4. \( h^{\epsilon}_\perp := \mathcal{S}^{\epsilon}((\mathbf{1} - \pi^{\epsilon}) H^{\epsilon}(\mathbf{1} - \pi^{\epsilon})) \);
5. \( h^{\epsilon}_* := \mathcal{S}^{\epsilon,k}(\pi^{\epsilon,k} H^{\epsilon,k}(\mathbf{1} - \pi^{\epsilon,k})) \),
6. \( h^{\epsilon}_* := \mathcal{S}^{\epsilon}(\pi^{\epsilon} H^{\epsilon}(\mathbf{1} - \pi^{\epsilon})) \);
7. \( r^{\epsilon,k} := \mathcal{S}^{\epsilon,k}((\mathbf{1} - \pi^{\epsilon,k}) R^{\epsilon,k}_\perp (\mathbf{1} - \pi^{\epsilon,k})) \),
8. \( r^{\epsilon} := \mathcal{S}^{\epsilon}((\mathbf{1} - \pi^{\epsilon}) R^{\epsilon}_\perp (\mathbf{1} - \pi^{\epsilon})) \);
9. \( \eta^{\epsilon,k} := \mathcal{S}^{\epsilon,k}(\pi^{\epsilon,k}(Y^{\epsilon,k})^{-1/2} \pi^{\epsilon,k}) \),
10. \( \eta^{\epsilon} := \mathcal{S}^{\epsilon}(\pi^{\epsilon}(Y^{\epsilon})^{-1/2} \pi^{\epsilon}) \);
11. \( \delta^{\epsilon,k} := \eta^{\epsilon,k} - p^{\epsilon,k} \),
12. \( \delta^{\epsilon} := \eta^{\epsilon} - p^{\epsilon} \).

**Proposition 8.4.** We have the following properties:

1. \( \{r^{\epsilon,k}, r^{\epsilon}\} \subset S^{-2}(\Xi) \),
2. \( \{\eta^{\epsilon,k}, \delta^{\epsilon,k}, \eta^{\epsilon}, \delta^{\epsilon}\} \subset S^{-\infty}(\Xi) \).

**Proof.** For the first two conclusions we just use Proposition 7.1 with \( m = 2 \). For the third conclusion we notice that \( \pi^{\epsilon,k} \) and \( \pi^{\epsilon} \) have symbols of class \( S^{-\infty}(\Xi) \). \( \square \)
8.2. Main approximation

8.2.1. Reduction to the constant field $\epsilon B_0$. The next proposition states that $\tilde{H}^{\epsilon,k}$ is close in operator norm to a magnetic matrix in which the non-constant magnetic field appears only through the phase $\Lambda^{\epsilon,k}$.

**Proposition 8.5.** With the definitions and notations of this section, the following approximation holds, for any $(\alpha, \beta) \in \Gamma \times \Gamma$,

$$
\langle \phi^\epsilon_{\alpha}, \tilde{H}^{\epsilon,k} \phi^\epsilon_{\beta} \rangle_{\mathcal{H}} = \Lambda^{\epsilon,k}(\alpha, \beta) \langle \phi^\epsilon_{\alpha}, \mathcal{D} p^\epsilon(h^\epsilon) \phi^\epsilon_{\beta} \rangle_{\mathcal{H}} + \kappa \epsilon \mathcal{O}(|\alpha - \beta|^{-\infty}).
$$

with

$$
h^\epsilon := h^\epsilon_0 + h^\epsilon_0 \mu^\epsilon \zeta^\epsilon + h^\epsilon_0 \mu^\epsilon \eta^\epsilon + \mu^\epsilon h^\epsilon_0 \mu^\epsilon \eta^\epsilon.
$$

Moreover

$$
\langle \phi^\epsilon_{\alpha}, \mathcal{D} p^\epsilon(h^\epsilon) \phi^\epsilon_{\beta} \rangle_{\mathcal{H}} = \Lambda^\epsilon(\alpha, \beta) \langle \phi^\epsilon_{\alpha-\beta}, \mathcal{D} p^\epsilon(h^\epsilon_0) \phi^\epsilon_{\beta} \rangle_{\mathcal{H}}. \tag{8.4}
$$

**Proof.** Having in mind the notation of Section 6 (see Definition 6.1, item 5), we get

$$
\langle \phi^\epsilon_{\alpha}, \pi^\epsilon \widetilde{H}^{\epsilon,k} \pi^\epsilon \phi^\epsilon_{\beta} \rangle_{\mathcal{H}} = \Sigma_1(\alpha, \beta) + \Sigma_2(\alpha, \beta), \tag{8.5}
$$

where, using (4.4) we can write

$$
\Sigma_1(\alpha, \beta) := \sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \mathbb{F}^{\epsilon,k}_{\alpha' \alpha} \mathbb{F}^{\epsilon,k}_{\beta' \beta} (Y^{\epsilon,k})^{-\frac{1}{2}} \Lambda^{\epsilon,k}(\cdot, \alpha') \tau_{\alpha'} \psi^\epsilon_0,
$$

$$
H^{\epsilon,k}(Y^{\epsilon,k})^{-\frac{1}{2}} \Lambda^{\epsilon,k}(\cdot, \beta') \tau_{\beta'} \psi^\epsilon_0 \rangle_{\mathcal{H}} \tag{8.6}
$$

and

$$
\Sigma_2(\alpha, \beta) := \sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \mathbb{F}^{\epsilon,k}_{\alpha' \alpha} \mathbb{F}^{\epsilon,k}_{\beta' \beta} (Y^{\epsilon,k})^{-\frac{1}{2}} \Lambda^{\epsilon,k}(\cdot, \alpha') \tau_{\alpha'} \psi^\epsilon_0,
$$

$$
K^{\epsilon,k}(Y^{\epsilon,k})^{-\frac{1}{2}} \Lambda^{\epsilon,k}(\cdot, \beta') \tau_{\beta'} \psi^\epsilon_0 \rangle_{\mathcal{H}}, \tag{8.7}
$$

with

$$
K^{\epsilon,k} := \pi^\epsilon \mathbb{H}^{\epsilon,k} R^{\perp\epsilon,k}_\perp(0) \mathbb{H}^{\epsilon,k} \pi^\epsilon \in \mathcal{L}(\pi^\epsilon \mathcal{H}).
$$

Using Formula (6.2) and Proposition 6.2 (point 3), we will express the series appearing in $\Sigma_1(\alpha, \beta)$ and $\Sigma_2(\alpha, \beta)$ in terms of elements associated with $\pi^\epsilon \mathcal{H}$ in the constant magnetic field $\epsilon B_0$.

In Subsection 8.2.2 we will prove the following estimation contained in Lemma 8.6:

$$
\Sigma_1(\alpha, \beta)
$$

$$
= \Lambda^{\epsilon,k}(\alpha, \beta) \langle \phi^\epsilon_{\alpha}, [H^{\epsilon} + \mathcal{D} p^\epsilon(h^\epsilon_0 \mu^\epsilon \zeta^\epsilon + h^\epsilon_0 \mu^\epsilon \eta^\epsilon + \mu^\epsilon h^\epsilon_0 \mu^\epsilon \eta^\epsilon)] \phi^\epsilon_{\beta} \rangle_{\mathcal{H}} + \kappa \epsilon \mathcal{O}(|\alpha - \beta|^{-\infty}).
$$
For the term $\Sigma_2(\alpha, \beta)$ we will prove in Subsection 8.2.3 (Lemma 8.11) that

$$
\Sigma_2(\alpha, \beta) = \tilde{\Lambda}^{\epsilon, \kappa}(\alpha, \beta) \langle \phi_\alpha^\epsilon, \Omega p^\epsilon(\psi^{\epsilon, \kappa} h_{\epsilon, \kappa}^\epsilon, \kappa r_{\epsilon, \kappa} h_{\epsilon, \kappa}^\epsilon, \kappa \psi^{\epsilon, \kappa}) \phi_\beta^\epsilon \rangle_{\mathfrak{g}} + \kappa \epsilon \mathcal{O}(|\alpha - \beta|^{-\infty}).
$$

We continue by noticing that Proposition 6.4 and the properties of the magnetic composition of symbols (see Proposition B.12 in [4]) imply that for any seminorm $v$ on $S^{-\infty}(\Xi)$ there exists $C_v > 0$ such that

$$
v(h_o^{\epsilon, \kappa} - h_o^\epsilon) \leq C_v \kappa \epsilon, \quad v(h_o^{\epsilon, \kappa} - h_o^\epsilon) \leq C_v \kappa \epsilon, \quad v(h_1^{\epsilon, \kappa} - h_1^\epsilon) \leq C_v \kappa \epsilon.
$$

(8.8)

Then let us consider the difference $r_{\epsilon, \kappa} - r^\epsilon \in S^{-2}(\Xi)$ and compute

$$
[h_{\epsilon, \kappa}^{\epsilon, \kappa}(r_{\epsilon, \kappa} - r^\epsilon)]^{\#} h_1^\epsilon = (1 - p^{\epsilon, \kappa})^{\#} h_1^\epsilon - (h_{\epsilon, \kappa}^{\epsilon, \kappa}(r_{\epsilon, \kappa} - r^\epsilon))^{\#} h_1^\epsilon.
$$

(8.9)

Using Propositions B.12 and B.14 in [4] and the estimates in Proposition 6.4 we conclude that

$$
(h_{\epsilon, \kappa}^{\epsilon, \kappa}(r_{\epsilon, \kappa} - r^\epsilon))^{\#} h_1^\epsilon = (h_{\epsilon, \kappa}^{\epsilon, \kappa}(r_{\epsilon, \kappa}))^{\#} h_1^\epsilon + \kappa \epsilon \tau_{1, \epsilon, \kappa} = h_{\epsilon, \kappa}^{\epsilon, \kappa}(1 - p^\epsilon) + \kappa \epsilon \tau_{1, \epsilon, \kappa},
$$

(8.10)

and

$$
[h_{\epsilon, \kappa}^{\epsilon, \kappa}(r_{\epsilon, \kappa} - r^\epsilon))^{\#} h_1^\epsilon
= (1 - p^{\epsilon, \kappa})^{\#} h_1^\epsilon - h_{\epsilon, \kappa}^{\epsilon, \kappa}(1 - p^\epsilon) - \kappa \epsilon \tau_{1, \epsilon, \kappa}
= h_1^\epsilon - h_{\epsilon, \kappa}^{\epsilon, \kappa} + \kappa \epsilon \tau_{2, \epsilon, \kappa}
= (1 - p^\epsilon)^{\#} h_1^\epsilon (1 - p^\epsilon) - (1 - p^{\epsilon, \kappa})^{\#} h_{\epsilon, \kappa}^{\epsilon, \kappa}(1 - p^\epsilon) - \kappa \epsilon \tau_{3, \epsilon, \kappa},
$$

with $\tau_{1, \epsilon, \kappa}$, $\tau_{2, \epsilon, \kappa}$ and $\tau_{3, \epsilon, \kappa}$ in $S^0(\Xi)$ uniformly for $(\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1]$.

Using Proposition 6.4 several times we obtain that the difference $r_{\epsilon, \kappa} - r^\epsilon$ belongs to $S^{-4}(\Xi)$ and for any seminorm $v$ on $S^{-4}(\Xi)$ there exists $C_v > 0$ such that

$$
v(r_{\epsilon, \kappa} - r^\epsilon) \leq C_v \kappa \epsilon.
$$

(8.11)

From Proposition 6.4 and the magnetic version of the Calderon–Vaillancourt Theorem (see Theorem 3.1 in [16]) we deduce that

$$
\| \Omega p^\epsilon(p^{\epsilon, \kappa} - p^\epsilon) \| \leq C \kappa \epsilon.
$$

(8.12)

Meanwhile, from Proposition B.14 in [4], we also deduce that for any symbols $f$ and $g$ in $S^0(\Xi)$ there exists $C > 0$ such that

$$
\| \Omega p^\epsilon(f^{\epsilon, \kappa} g - f^{\#} g) \| \leq C \kappa \epsilon.
$$

(8.13)
Putting all these results together, using (8.8) and (8.11) in order to control the difference \( \tilde{\delta}^{\varepsilon} - \delta^{\varepsilon} \) and the usual result concerning the estimate of the difference of the two magnetic products \( \#^{\varepsilon, K} \) and \( \#^{\varepsilon} \) (see Proposition B.14 in [4]) we finally obtain

\[
\Sigma_1(\alpha, \beta) = \tilde{\Lambda}^{\varepsilon, K}(\alpha, \beta) \langle \phi^\varepsilon_{\alpha \cdot \beta}, \Omega p^\varepsilon (h^{\varepsilon}_{\alpha} + h^{\varepsilon}_{\alpha} \#^{\varepsilon} \delta^{\varepsilon} + \delta^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \eta^{\varepsilon}) \phi^\varepsilon_{\beta} \rangle_{\gamma \varepsilon} + \kappa \varepsilon O(|\alpha - \beta|^{-\infty}),
\]

(8.14)

and

\[
\Sigma_2(\alpha, \beta) = \tilde{\Lambda}^{\varepsilon, K}(\alpha, \beta) \langle \phi^\varepsilon_{\alpha \cdot \beta}, \Omega p^\varepsilon (\eta^{\varepsilon} \#^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \delta^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon}) \phi^\varepsilon_{\beta} \rangle_{\gamma \varepsilon} + \kappa \varepsilon O(|\alpha - \beta|^{-\infty}).
\]

(8.15)

This gives us (8.3) in the proposition.

From Proposition 8.2 it follows that the symbol

\[
h^\varepsilon := h^{\varepsilon}_{\alpha} + h^{\varepsilon}_{\alpha} \#^{\varepsilon} \delta^{\varepsilon} + \delta^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \eta^{\varepsilon} + \eta^{\varepsilon} \#^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon}
\]

(8.16)

is \( \Gamma \)-periodic in the \( \chi \)-variable. Thus, using the results in Proposition 8.1, \( \Omega p^\varepsilon (h^\varepsilon) \) commutes with all the operators \( \{ \Lambda^\varepsilon (\gamma, \cdot) \tau_{\gamma^c} \}_{\gamma \in \Gamma} \) and we get (8.4), achieving the proof of the proposition.

**8.2.2. Control of \( \Sigma_1(\alpha, \beta) \).** The main result here is the following lemma.

**Lemma 8.6.** We have

\[
\Sigma_1(\alpha, \beta) = \tilde{\Lambda}^{\varepsilon, K}(\alpha, \beta) \langle \phi^\varepsilon_{\alpha \cdot \beta}, [H^{\varepsilon} + \Omega p^\varepsilon (h^{\varepsilon}_{\alpha} \#^{\varepsilon} \#^{\varepsilon} \delta^{\varepsilon} + \#^{\varepsilon} h^{\varepsilon}_{\alpha} \#^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon} \eta^{\varepsilon}) \phi^\varepsilon_{\beta} \rangle_{\gamma \varepsilon} + \kappa \varepsilon O(|\alpha - \beta|^{-\infty}).
\]

The rest of this paragraph is dedicated to the proof of the above statement. We start by considering the scalar products appearing in the double series in the expression of \( \Sigma_1(\alpha, \beta) \) in (8.6).

\[
S_{\alpha' \beta'}^{\varepsilon, K} := \langle (Y^{\varepsilon, K})^{-1/2} \Lambda^{\varepsilon, K} (\cdot, \alpha'), \tau_{\alpha'} \psi_0^\varepsilon, H^{\varepsilon, K} (Y^{\varepsilon, K})^{-1/2} \Lambda^{\varepsilon, K} (\cdot, \beta') \tau_{-\beta'} \psi_0^\varepsilon \rangle_{\gamma \varepsilon} + \tilde{S}_1(\alpha', \beta') + \tilde{S}_2(\alpha', \beta'),
\]

(8.17)

with

\[
\tilde{S}_1(\alpha', \beta') := \langle \phi^\varepsilon_{\alpha' \cdot \beta'} \Lambda^{\varepsilon, K} (\alpha', \cdot) H^{\varepsilon, K} (Y^{\varepsilon, K})^{-1/2} - 1 \rangle \Lambda^{\varepsilon, K} (\cdot, \beta') \phi^\varepsilon_{\beta'}, \gamma \varepsilon \rangle
\]

(8.18)

and

\[
\tilde{S}_2(\alpha', \beta') := \langle \phi^\varepsilon_{\alpha' \cdot \beta'} \Lambda^{\varepsilon, K} (\alpha', \cdot) [(Y^{\varepsilon, K})^{-1/2} - 1] H^{\varepsilon, K} (Y^{\varepsilon, K})^{-1/2} \Lambda^{\varepsilon, K} (\cdot, \beta') \phi^\varepsilon_{\beta'}, \gamma \varepsilon \rangle
\]

(8.19)
Lemma 8.7. Let $m \in \mathbb{N}$. There exists $C_m > 0$ such that for all $\alpha', \beta' \in \Gamma$ and $0 < \varepsilon, \kappa \leq 1$,
\[
|\langle \phi_{\alpha'}^\varepsilon, \bar{\Lambda}^{\varepsilon,k}(\alpha', \cdot)H^{\varepsilon,k}(\cdot, \beta')\phi_{\beta'}^\varepsilon \rangle_{\mathcal{H}} - \Lambda^{\varepsilon,k}(\alpha', \beta')\langle \phi_{\alpha'}^\varepsilon, H^{\varepsilon}(\cdot)\phi_{\beta'}^\varepsilon \rangle_{\mathcal{H}}| \leq C_m \kappa \varepsilon (\alpha' - \beta')^{-m}.
\]

Proof. We use (5.7) for the pair of vector potentials $(A^{\varepsilon,k}, A^{\varepsilon})$, in the form
\[
(-i \nabla - A^{\varepsilon,k}(x))^2 \Lambda^{\varepsilon,k}(x, \beta) = \Lambda^{\varepsilon,k}(x, \beta)(-i \nabla - A^{\varepsilon}(x) + \kappa a_\varepsilon(x, \beta))^2,
\]
writing
\[
a_\varepsilon(x, z) = \sum_k (x - \gamma)_k \int_0^1 \epsilon B_{jk}(\epsilon \gamma + s \epsilon(x - \gamma))ds \text{ for } j = 1, 2,
\]
and noticing that
\[
|a_\varepsilon(x, \gamma)| \leq C \epsilon |x - \gamma|.
\]
Using (5.3) we have
\[
\Lambda^{\varepsilon,k}(x, \alpha)^{-1} \Lambda^{\varepsilon,k}(x, \beta) = \Lambda^{\varepsilon,k}(\alpha, \beta)\bar{\Gamma}^{\varepsilon,k}(\alpha, x, \beta),
\]
and we know from (5.2) and (5.4) that
\[
|\bar{\Gamma}^{\varepsilon,k}(\alpha, x, \beta) - 1| \leq C \kappa \epsilon |\alpha - \alpha||x - \beta|.
\]

The fast decay of the quasi Wannier function (Proposition 3.3) allows us to finish the proof.

Lemma 8.8. Let us consider an operator of the form $\pi^{\varepsilon,k} \mathcal{D}^{\varepsilon,k}(F)\pi^{\varepsilon,k}$ with $F \in S_1^-(\Xi)$ (see (5.10)). Then for any $m \in \mathbb{N}$, there exists a seminorm $\nu_m$ such that, for all $(\alpha', \beta') \in \Gamma \times \Gamma$ and $(\epsilon, \kappa) \in [0, 1] \times [0, 1]$,
\[
|\langle \phi_{\alpha'}^\varepsilon, \bar{\Lambda}^{\varepsilon,k}(\alpha', \cdot)\pi^{\varepsilon,k}(F)\pi^{\varepsilon,k}(\cdot, \beta')\phi_{\beta'}^\varepsilon \rangle_{\mathcal{H}} - \Lambda^{\varepsilon,k}(\alpha', \beta')\langle \phi_{\alpha'}^\varepsilon, \pi^{\varepsilon}(F)\pi^{\varepsilon}(\cdot)\phi_{\beta'}^\varepsilon \rangle_{\mathcal{H}}| \leq \nu_m(F) \kappa \varepsilon (\alpha' - \beta')^{-m}.
\]
Proof. Using Proposition B.8 in [4] we may conclude that for \( F \in S_1^{-}(\Xi) \) and for any \( k \in \mathbb{N} \) there exists a seminorm \( v_k : S_1^{-}(\Xi) \to \mathbb{R}_+ \) such that

\[
\sup_{x \in X} \int |x - y|^k |K_F(x, y) dy| \leq v_k(F). \tag{8.23}
\]

Let us compute for \( u \in \mathcal{H} \):

\[
[\tilde{\Lambda}^{\varepsilon, \kappa}(\alpha', \cdot) Dp^{\varepsilon, \kappa}(F) \tilde{\Lambda}^{\varepsilon, \kappa}(\cdot, \beta') u](x) = [\tilde{\Lambda}^{\varepsilon, \kappa}(\alpha', \cdot) [\text{Int}(\tilde{\Lambda}^{\varepsilon, \kappa} \Lambda^\varepsilon K_F)(\tilde{\Lambda}^{\varepsilon, \kappa}(\cdot, \beta') u)]](x)
= \tilde{\Lambda}^{\varepsilon, \kappa}(\alpha', \beta') \tilde{\Omega}^{\varepsilon, \kappa}(\alpha', x, \beta') \int dy \tilde{\Omega}^{\varepsilon, \kappa}(x, y, \beta') \Lambda^\varepsilon(x, y) K_F(x, y) u(y), \tag{8.24}
\]

and take into account the estimates (from (5.4))

\[
|\tilde{\Omega}^{\varepsilon, \kappa}(\alpha', x, \beta') - 1| \leq C \kappa \varepsilon |x - \alpha'| |x - \beta'|,
\]

and

\[
|\tilde{\Omega}^{\varepsilon, \kappa}(x, x + z, \beta') - 1| \leq C \kappa \varepsilon |z||x - \beta'|.
\]

Thus we obtain

\[
|\langle \phi_{\alpha'}^{\varepsilon, \kappa}, \tilde{\Lambda}^{\varepsilon, \kappa}(\alpha', \cdot) \pi^{\varepsilon, \kappa} Dp^{\varepsilon, \kappa}(F) \pi^{\varepsilon, \kappa} \tilde{\Lambda}^{\varepsilon, \kappa}(\cdot, \beta') \phi_{\beta'}^{\varepsilon, \kappa} \rangle_{\mathcal{H}^\varepsilon, \kappa}
- \tilde{\Lambda}^{\varepsilon, \kappa}(\alpha, \beta) \langle \phi_{\alpha}^{\varepsilon, \kappa}, \pi^{\varepsilon} Dp^{\varepsilon}(F) \pi^{\varepsilon} \phi_{\beta}^{\varepsilon} \rangle_{\mathcal{H}^\varepsilon, \kappa}|
\leq \kappa \varepsilon v_m(F) |\alpha' - \beta'|^m,
\]

for any \( m \in \mathbb{N} \), with some \( v_m : S_0^0(\Xi) \to \mathbb{R}_+ \) a seminorm on \( S_0^0(\Xi) \). \( \square \)

**Lemma 8.9.** For any \( m \in \mathbb{N} \) there exists \( C_m > 0 \) such that

\[
|\tilde{\Omega}^{\varepsilon, \kappa}_{\alpha', \beta'}| \leq C_m (\alpha' - \beta')^{-m}.
\]

**Proof.** We have

\[
\tilde{\Omega}^{\varepsilon, \kappa}_{\alpha', \beta'} = \langle (Y^{\varepsilon, \kappa})^{-1/2} \Lambda^{\varepsilon, \kappa}(\cdot, \alpha') \tau_{\alpha'} \psi_{\alpha}^{\varepsilon, \kappa}, H^{\varepsilon, \kappa}(Y^{\varepsilon, \kappa})^{-1/2} \Lambda^{\varepsilon, \kappa}(\cdot, \beta') \tau_{-\beta'} \psi_{\beta}^{\varepsilon, \kappa} \rangle_{\mathcal{H}^\varepsilon, \kappa}
= \langle \phi_{\alpha'}^{\varepsilon, \kappa}, Dp^{\varepsilon, \kappa}(\eta^{\varepsilon, \kappa, \eta^{\varepsilon, \kappa}}_\# h_{\alpha'}^{\varepsilon, \kappa} \eta^{\varepsilon, \kappa} h_{\beta'}^{\varepsilon, \kappa} \eta^{\varepsilon, \kappa}) \phi_{\beta'}^{\varepsilon, \kappa} \rangle_{\mathcal{H}^\varepsilon, \kappa}.
\]
We notice that \( h_{\varepsilon, \kappa} \varphi_{\alpha} \in S^{-\infty}(\mathbb{X}) \) so that using once again Proposition B.8 from [4] as in the beginning of the proof of Lemma 8.8 and the rapid decay of the magnetic distorted Wannier functions we conclude that for any \( m \in \mathbb{N} \) there exist \( C_m > 0, C'_m > 0 \) and a seminorm \( v_m \) on \( S^{-\infty}(\mathbb{X}) \), such that

\[
|\tilde{\mathcal{S}}_{\alpha', \beta'}^{(\varepsilon, \kappa)}| \\
\leq C'_m v_m(\eta^{\varepsilon, \kappa}, h_{\varepsilon, \kappa} \varphi_{\alpha}) \int_{\mathbb{X}} dx dy \langle x - \alpha' \rangle^{-m-3} \langle x - y \rangle^{-m} \langle y - \beta' \rangle^{-m-3} \\
\leq C_m (\alpha' - \beta')^{-m}.
\]

\[\square\]

**Remark 8.10.** Let us notice that

\[
\sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \mathcal{F}_\alpha \mathcal{F}_{\alpha'}^{(\varepsilon, \kappa)} H_{\alpha, \beta} \mathcal{S}_{\alpha', \beta'}^{(\varepsilon, \kappa)} \\
= \mathcal{S}_{\alpha, \beta}^{(\varepsilon, \kappa)} + \sum_{\alpha' \in \Gamma} (\mathcal{F}_{\alpha} - 1) \mathcal{P}_{\alpha}^{(\varepsilon, \kappa)} + \sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \mathcal{F}_{\alpha'} \mathcal{F}_{\beta'}^{(\varepsilon, \kappa)} \mathcal{S}_{\alpha', \beta'}^{(\varepsilon, \kappa)}
\]

and due to Proposition 6.2 (point 3) we deduce that for any \((\varepsilon, \kappa) \in [0, \varepsilon_0] \times [0, 1]\) for some small enough \( \varepsilon_0 > 0 \) and for any \((\alpha, \beta) \in \Gamma \times \Gamma\) we have that for any \( m \in \mathbb{N} \) there exists some \( C_m > 0 \) such that

\[
\left| \sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \mathcal{F}_{\alpha'} \mathcal{F}_{\beta'}^{(\varepsilon, \kappa)} \mathcal{S}_{\alpha', \beta'}^{(\varepsilon, \kappa)} - \mathcal{S}_{\alpha, \beta}^{(\varepsilon, \kappa)} \right| \leq C_m \varepsilon (\alpha - \beta)^{-m}. \quad (8.25)
\]

Coming to the last two terms (8.18) and (8.19) we notice that they may be written as

\[
S_1 = \langle \phi_{\alpha'}^{(\varepsilon, \kappa)}, \tilde{\mathcal{L}}^{(\varepsilon, \kappa)}(\alpha', \cdot) \rangle \mathcal{P}^{(\varepsilon, \kappa)}(h_{\varepsilon, \kappa} \delta^{(\varepsilon, \kappa)} \mathcal{S}_{\alpha, \beta}^{(\varepsilon, \kappa)} \mathcal{L}^{(\varepsilon, \kappa)}(\cdot, \beta') \phi_{\beta'}^{(\varepsilon, \kappa)} \rangle_{\mathcal{C}},
\]

and

\[
S_2 = \langle \phi_{\alpha'}^{(\varepsilon, \kappa)}, \tilde{\mathcal{L}}^{(\varepsilon, \kappa)}(\alpha', \cdot) \rangle \mathcal{P}^{(\varepsilon, \kappa)}(h_{\varepsilon, \kappa} \delta^{(\varepsilon, \kappa)} \mathcal{S}_{\alpha, \beta}^{(\varepsilon, \kappa)} \mathcal{L}^{(\varepsilon, \kappa)}(\cdot, \beta') \phi_{\beta'}^{(\varepsilon, \kappa)} \rangle_{\mathcal{C}}.
\]

Thus we notice that \( h_{\varepsilon, \kappa} \delta^{(\varepsilon, \kappa)} \) and \( \delta^{(\varepsilon, \kappa)} h_{\varepsilon, \kappa} \eta^{(\varepsilon, \kappa)} \) are symbols of class \( S^{-\infty}(\mathbb{X}) \) and we may use Lemma 8.8 in order to obtain that

\[
|\langle \phi_{\alpha'}^{(\varepsilon, \kappa)}, \tilde{\mathcal{L}}^{(\varepsilon, \kappa)}(\alpha', \cdot) \mathcal{P}^{(\varepsilon, \kappa)}(h_{\varepsilon, \kappa} \delta^{(\varepsilon, \kappa)} \mathcal{S}_{\alpha, \beta}^{(\varepsilon, \kappa)} \mathcal{L}^{(\varepsilon, \kappa)}(\cdot, \beta') \phi_{\beta'}^{(\varepsilon, \kappa)} \rangle_{\mathcal{C}}| \\
\leq C_m \varepsilon (\alpha' - \beta')^{-m}.
\]
and

\[
\begin{align*}
\left| \langle \phi_{\alpha'}^\epsilon, \tilde{\Lambda}^{\epsilon, K}(\alpha', \cdot) | (Y^{\epsilon, K})^{-1/2} - 1 \right| H^{\epsilon, K}(Y^{\epsilon, K})^{-1/2} & \tilde{\Lambda}^{\epsilon, K}(\cdot, \beta') \phi_{\beta'}^\epsilon \rangle \mathcal{J}_\ell \\
- \langle \phi_{\alpha'}^\epsilon, \mathcal{D}p^\epsilon (\delta^{\epsilon, K}_h, h^{\epsilon, K}_h, h^{\epsilon, K}_h, \eta^{\epsilon, K}) \phi_{\beta'}^\epsilon \rangle \mathcal{J}_\ell \right| \\
\leq C m \kappa \epsilon (\alpha' - \beta')^{-m}.
\end{align*}
\]

8.2.3. Control of $\Sigma_2(\alpha, \beta)$. Going back to the double series appearing in (8.7) we write

\[
\mathcal{R}^{\epsilon, K}_{\alpha', \beta'} = (Y^{\epsilon, K})^{-1/2} \tilde{\Lambda}^{\epsilon, K}(\cdot, \alpha') \tau_{\alpha'} \psi_0^{\epsilon} \pi^{\epsilon, K}_h H^{\epsilon, K}_h \tilde{R}^{\epsilon, K}_h(0) H^{\epsilon, K}_h \pi^{\epsilon, K}_h (Y^{\epsilon, K})^{-1/2} \tilde{\Lambda}^{\epsilon, K}(\cdot, \beta') \tau_{\beta'} \psi_0^{\epsilon} \mathcal{J}_\ell
\]

\[
= \langle \phi_{\alpha'}^{\epsilon}, \tilde{\Lambda}^{\epsilon, K}(\alpha', \cdot) (Y^{\epsilon, K})^{-1/2} \pi^{\epsilon, K}_h H^{\epsilon, K}_h \tilde{R}^{\epsilon, K}_h(0) H^{\epsilon, K}_h \pi^{\epsilon, K}_h (Y^{\epsilon, K})^{-1/2} \tilde{\Lambda}^{\epsilon, K}(\cdot, \beta') \phi_{\beta'}^{\epsilon} \rangle \mathcal{J}_\ell
\]

\[
= \langle \phi_{\alpha'}^{\epsilon}, \tilde{\Lambda}^{\epsilon, K}(\alpha', \cdot) \mathcal{D}p^{\epsilon, K}(\eta^{\epsilon, K}_h, h^{\epsilon, K}_h, h^{\epsilon, K}_h, \eta^{\epsilon, K}) \tilde{\Lambda}^{\epsilon, K}(\cdot, \beta') \phi_{\beta'}^{\epsilon} \rangle \mathcal{J}_\ell.
\]

and notice that

\[
\sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \tilde{F}_{\alpha' \alpha} \tilde{F}_{\beta' \beta} \mathcal{R}^{\epsilon, K}_{\alpha', \beta'}
\]

\[
= \mathcal{R}^{\epsilon, K}_{\alpha, \beta} + \sum_{\alpha' \in \Gamma} (\tilde{F}_{\alpha' \alpha} - 1) \mathcal{R}^{\epsilon, K}_{\alpha', \beta} + \sum_{\alpha' \in \Gamma} \sum_{\beta' \in \Gamma} \tilde{F}_{\alpha' \alpha} (\tilde{F}_{\epsilon, K} - 1) \beta' \beta' \mathcal{R}^{\epsilon, K}_{\alpha', \beta'}.
\]

Noticing further that $h^{\epsilon, K}_h := p^{\epsilon, K}_h h^{\epsilon, K}_h, h^{\epsilon, K}_h, (1 - p^{\epsilon, K}_h) \in \mathbb{S}^{-\infty}(\mathbb{Z})$ and using again Lemma 8.8 we obtain that for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

\[
\left| \mathcal{R}^{\epsilon, K}_{\alpha', \beta'} - \tilde{\Lambda}^{\epsilon, K}(\cdot, \beta') \langle \phi_{\alpha'}^{\epsilon}, \mathcal{D}p^\epsilon (\eta^{\epsilon, K}_h, h^{\epsilon, K}_h, h^{\epsilon, K}_h, \eta^{\epsilon, K}) \phi_{\beta'}^{\epsilon} \rangle \mathcal{J}_\ell \right|
\]

\[
\leq C m \kappa \epsilon (\alpha' - \beta')^{-m}.
\]

Moreover the proof of Lemma 8.9 applies in this situation also and we obtain that for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

\[
\left| \mathcal{R}^{\epsilon, K}_{\alpha', \beta'} \right| \leq C m (\alpha' - \beta')^{-m}.
\]

Taking into account all these results ((8.26)-(8.28)) and point 3 of Proposition 6.2 we obtain the following result.

**Lemma 8.11.** We have

\[
\Sigma_2(\alpha, \beta) = \tilde{\Lambda}^{\epsilon, K}(\alpha, \beta) \langle \phi_{\alpha}^{\epsilon}, \mathcal{D}p^\epsilon (\eta^{\epsilon, K}_h, h^{\epsilon, K}_h, h^{\epsilon, K}_h, \eta^{\epsilon, K}) \phi_{\beta}^{\epsilon} \rangle \mathcal{J}_\ell
\]

\[+ \kappa \epsilon \mathcal{O}(\alpha - \beta)^{-\infty}.
\]
8.3. The modified energy band and proof of Theorem 2.2(i). Having in mind the result of Proposition 8.5 let us introduce

\[ k^e: \Gamma \longrightarrow Ck^e(\gamma) := \langle \phi^e_\alpha, \mathcal{D}p^e(h^e)\phi^e_\beta \rangle_{\mathcal{H}}; \quad \hat{x}^e(\theta) := \sum_{\gamma \in \Gamma} e^{-i(\beta, \gamma)}k^e(\gamma). \]  

(8.29)

Proceeding as in [8] and [4] we define the magnetic matrix acting in \( \ell^2(\Gamma) \),

\[ M^{e,\kappa}(\alpha, \beta) := \langle \phi^e_\alpha, \mathcal{D}p^e(\hbar^e)\phi^e_\beta \rangle_{\mathcal{H}} = \Lambda^{e,\kappa}(\alpha, \beta)k^e(\alpha - \beta). \]  

(8.30)

Let us consider the smooth function \( \hat{x}^e: \mathbb{T} \rightarrow \mathbb{R} \) as a periodic smooth function on \( \Xi \) constant along the directions in \( \Xi \times \{0\} \) and write its magnetic quantization as an integral operator

\[ \mathcal{D}p^{e,\kappa}(\hat{x}^e) := \operatorname{Int}(\Lambda^{e,\kappa}K_{\hat{x}^e}), \]  

(8.31)

with

\[ K_{\hat{x}^e}(x, y) = (2\pi)^{-2} \int_{\hat{x}^e} e^{i(\xi, x - y)}\hat{x}^e(\xi) d\xi = \sum_{\gamma \in \Gamma} k^e(\gamma)\delta_0(x - y - \gamma). \]  

(8.32)

Let us define the following unitary map

\[ \mathcal{W}_\Gamma: L^2(\Xi) \longrightarrow \ell^2(\Gamma) \otimes L^2(E)(\mathcal{W}_\Gamma F)(\alpha, x) := F(\alpha + x) \]  

(8.33)

and compute the integral kernel \( \mathcal{K} \) of \( \mathcal{W}_\Gamma(\mathcal{D}p^{e,\kappa}(\hat{x}^e))\mathcal{W}_\Gamma^{-1} \) in this representation

\[ \mathcal{K}(\alpha + x, \beta + y) = \Lambda^{e,\kappa}(\alpha + x, \beta + y)k^e(\alpha - \beta)\delta_0(x - y). \]  

(8.34)

In order to compare it with (8.30) we shall consider two unitary gauge transformations:

\[ \mathcal{U}_e: \ell^2(\Gamma) \otimes L^2(E) \longrightarrow \ell^2(\Gamma) \otimes L^2(E), \quad (\mathcal{U}_e \Phi)(\alpha, x) := \Lambda^e(x, \alpha)\Phi(\alpha, x) \]  

and

\[ \mathcal{U}_{e,\kappa}: \ell^2(\Gamma) \otimes L^2(E) \longrightarrow \ell^2(\Gamma) \otimes L^2(E), \]  

with

\[ (\mathcal{U}_{e,\kappa} \Phi)(\alpha, x) := \tilde{\Lambda}^{e,\kappa}(\alpha, \alpha + x)\Phi(\alpha, x). \]

Using also (5.3), the kernel \( \tilde{\mathcal{K}} \) of \( (\mathcal{U}_{e,\kappa} \mathcal{U}_e \mathcal{W}_\Gamma)(\mathcal{D}p^{e,\kappa}(\hat{x}^e))(\mathcal{U}_{e,\kappa} \mathcal{U}_e \mathcal{W}_\Gamma)^{-1} \) is given by

\[ \tilde{\mathcal{K}}((\alpha, x); (\beta, y)) = \tilde{\Lambda}^{e,\kappa}(\alpha, \alpha + x)\Lambda^e(x, \alpha)\Lambda^{e,\kappa}(\alpha + x, \beta + x) \]

\[ \Lambda^e(\beta, x)\tilde{\Lambda}^{e,\kappa}(\beta + x, \beta)k^e(\alpha - \beta)\delta_0(x - y) \]

\[ = \Lambda^{e,\kappa}(\alpha, \beta)\tilde{\Sigma}^{e,\kappa}(\alpha, \alpha + x, \beta + x)\tilde{\Omega}^{e,\kappa}(\alpha, \beta + x, \beta)k^e(\alpha - \beta)\delta_0(x - y). \]
Proposition 8.12. The Hausdorff distance between the spectra of the operator $\mathcal{D}p^{ε,k}(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$ and the hermitian operator associated with the matrix $\mathcal{M}^{ε,k}(\alpha, \beta)$ in the orthonormal basis $\{\phi^{ε,k}_{γ}\}_{γ \in Γ}$ of $\pi^{ε,k}\mathcal{H}$ is of order $κε$.

Proof. If we denote by $x := α + x$ and $x' := β + x$ and use (5.4) we obtain that

$$|1 - Ω^{ε,k}(α, x, x')| \leq C_1 |α - β| κε,$$

and

$$|1 - Ω^{ε,k}(α, x', β)| \leq C_2 |α - β| κε.$$

Taking into account the rapid decay of $k(γ)$ for $|γ| → ∞$ and considering the canonical orthonormal basis $\{e_γ\}_{γ ∈ Γ}$ of $ℓ^2(Γ)$ defined by $e_γ(α) := δ_{γ,α}$ we obtain

$$\|((U_{ε,k}U_ε W_Γ)(\mathcal{D}p^{ε,k}(\mathcal{H}))(U_{ε,k}U_ε W_Γ)^{-1}
- \sum_{(α, β) ∈ Γ × Γ} \mathcal{M}^{ε,k}(α, β)(|e_α⟩⟨e_β| \otimes 1_{L^2(E)})\|
\leq C_1 κε.$$

On the other hand, the two operators

$$\sum_{(α, β) ∈ Γ × Γ} \mathcal{M}^{ε,k}(α, β)(|e_α⟩⟨e_β| \otimes 1_{L^2(E)})$$

and

$$\tilde{H}^{ε,k} = \sum_{(α, β) ∈ Γ × Γ} \mathcal{M}^{ε,k}(α, β)|\phi^{ε,k}_α⟩⟨\phi^{ε,k}_β|$$

have the same spectrum. □

Putting together the Propositions 8.5, 8.12 and Proposition 3.19 in [4] we obtain the second conclusion of our Theorem 2.2.

8.4. Behavior of the modified energy band function in the chosen energy window and proof of Theorem 2.2(ii). We will only prove Theorem 2.2(ii) in the case when $m = 0$; the general case is similar due to the rapid decay of the quasi Wannier function $ψ_0$.

Proposition 8.13. For $b ∈ (0, \tilde{b})$ with $\tilde{b}$ as in Lemma 3.1 there exists $ε_0 > 0$ and $C > 0$ such that, for any $θ ∈ Σ_b$ and any $ε ∈ [0, ε_0]$,

$$|\hat{χ}^{ε}(θ) - λ_0(θ)| \leq C ε.$$
Proof. Let us also define the smooth function
\[
\hat{x}_\epsilon^\gamma(\theta) := \sum_{\gamma \in \Gamma} e^{-i(\theta, \gamma)} \langle \phi_\gamma^\epsilon, \partial \phi_0^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma\epsilon} \tag{8.35}
\]
and estimate the above difference by a two step procedure:
\[
\hat{x}_\epsilon^\gamma(\theta) - \lambda_0(\theta) = (\hat{x}_\epsilon(\theta) - \hat{x}_\epsilon^\gamma(\theta)) + (\hat{x}_\epsilon^\gamma(\theta) - \lambda_0(\theta)). \tag{8.36}
\]

**Step 1.** We begin by computing
\[
\hat{x}_\epsilon^\gamma(\theta) - \lambda_0(\theta) = \sum_{\gamma \in \Gamma} e^{-i(\theta, \gamma)} \langle \phi_\gamma^\epsilon, \partial \phi_0^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma\epsilon} - \lambda_0(\theta).
\]

Using Proposition 8.1 we obtain
\[
\langle \phi_\gamma^\epsilon, \partial \phi_0^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma\epsilon} = \langle \psi_\gamma, H^0 \psi_0 \rangle_{\gamma\epsilon} = \Lambda^\epsilon(\alpha, \beta) \langle \phi_{\alpha-\beta}^\epsilon, \partial \phi_0^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma\epsilon}.
\]

Taking also into account that we have fixed the gauge for $A_0$ as in (1.6), we conclude that
\[
\langle \phi_\gamma^\epsilon, \partial \phi_0^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma\epsilon} = \langle \psi_\gamma, H^0 \psi_0 \rangle_{\gamma\epsilon},
\]
\[
+ \epsilon [\langle \phi_\gamma^\epsilon, (D \cdot A_0(\cdot) + A_0(\cdot) \cdot D + \epsilon A_0(\cdot)^2) \psi_0^\epsilon \rangle_{\gamma\epsilon}]
+ [(\tau_\gamma (\psi_0^\epsilon - \psi_0)), H^0 \psi_0^\epsilon \rangle_{\gamma\epsilon} + \langle \tau_\gamma \psi_0^\epsilon, H^0 (\psi_0^\epsilon - \psi_0) \rangle_{\gamma\epsilon}]
+ \langle \tau_\gamma \psi_0^\epsilon, (\Omega^\epsilon(\gamma, 0, \cdot) - 1)((D + \epsilon a_0(\cdot, 0))^2 + V(\cdot)) \psi_0^\epsilon \rangle_{\gamma\epsilon}.
\]

Using Definition 6.1 that, for any $\gamma \in \Gamma$,
\[
\phi_\gamma^\epsilon = \Lambda^\epsilon(\cdot, \gamma) \tau_\gamma \sum_{\beta \in \Gamma} \Omega^\epsilon(\beta, 0, \cdot) F^\epsilon(\beta) \tau_\beta \psi_0.
\]

Taking into account the rapid decay of $F$ (see Proposition 6.2) and of $\psi_0$, we conclude that for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that, for any $\epsilon \in [0, \epsilon_0]$ and $\gamma \in \Gamma$,
\[
|\langle \phi_\gamma^\epsilon, (D \cdot A_0(\cdot) + A_0(\cdot) \cdot D + \epsilon A_0(\cdot)^2) \psi_0^\epsilon \rangle_{\gamma\epsilon}| \leq C_m \langle \gamma \rangle^{-m},
\]
and
\[
|\langle \tau_\gamma \psi_0^\epsilon, (\Omega^\epsilon(\gamma, 0, \cdot) - 1)((D + \epsilon a_0(\cdot))^2 + V(\cdot)) \psi_0^\epsilon \rangle_{\gamma\epsilon}| \leq C_m \langle \gamma \rangle^{-m}.
\]

Using Definition 6.1 and point (2) in Proposition 6.2 we get also that for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that, for any $\epsilon \in [0, \epsilon_0]$ and $\gamma \in \Gamma$,
\[
|\langle (\tau_\gamma (\psi_0^\epsilon - \psi_0)), H^0 \psi_0^\epsilon \rangle_{\gamma\epsilon} + \langle \tau_\gamma \psi_0^\epsilon, H^0 (\psi_0^\epsilon - \psi_0) \rangle_{\gamma\epsilon}| \leq C_m \langle \gamma \rangle^{-m}.
\]
We have thus obtained
\[ \hat{\mathcal{E}}(\theta) - \lambda_0(\theta) = \sum_{\gamma \in \Gamma} e^{-i(\theta,\gamma)} \langle \psi_0, H^0 \psi_0 \rangle_{\gamma} - \lambda_0(\theta) + O(\epsilon). \]

Let us now use the Bloch–Floquet representation and the properties of \( \hat{\psi}_0 \) as constructed in Section 3 in order to write
\[ \langle \psi_0, H^0 \psi_0 \rangle_{\gamma} = \int_{\mathbb{T}_x} d\omega e^{i(\omega,\gamma)} \left( \langle \hat{\psi}_0(\omega), \left( \sum_{n \geq 0} \lambda_n(\omega) \langle \hat{\phi}_n(\omega) \rangle \right) \hat{\psi}_0(\omega) \right)_{\gamma}, \]
and we notice that for \( \omega \in \Sigma_b \) we have \( \hat{\psi}_0(\omega) = \hat{\phi}_0(\omega) \). Finally we can write
\[ \sum_{\gamma \in \Gamma} e^{-i(\theta,\gamma)} \langle \psi_0, H^0 \psi_0 \rangle_{\gamma} = \sum_{n \in \mathbb{N}} \langle \hat{\psi}_0(\theta), \hat{\phi}_n(\theta) \rangle_{\gamma}^2, \]
and thus
\[ \sum_{\gamma \in \Gamma} e^{-i(\theta,\gamma)} \langle \psi_0, H^0 \psi_0 \rangle_{\gamma} = \lambda_0(\theta) \text{ for } \theta \in \Sigma_b. \]

**Step 2.** We have to study the difference
\[ \hat{\mathcal{E}}(\theta) - \hat{\mathcal{E}}_0(\theta) = \sum_{\gamma \in \Gamma} e^{-i(\theta,\gamma)} \left( \langle \phi_0^\epsilon, \mathcal{D}p^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma} - \langle \phi_0^\epsilon, \mathcal{D}p^\epsilon (h_0^\epsilon) \phi_0^\epsilon \rangle_{\gamma} \right). \]

Thus let us analyse the symbol
\[ h^\epsilon - h_0^\epsilon = (h_0^\epsilon \# \delta^\epsilon + \delta^\epsilon \# h_0^\epsilon) + \delta^\epsilon \# h_0^\epsilon \# \delta^\epsilon + \eta^\epsilon \# h_0^\epsilon \# \eta^\epsilon + \eta^\epsilon \# \delta^\epsilon \# h_0^\epsilon \# \eta^\epsilon, \]
where \( \eta^\epsilon \) and \( \delta^\epsilon \) are defined in the last item in Definition 8.3.

Let \( h_o \in S^{-\infty}(\mathbb{Z}) \) be the symbol of the bounded operator
\[ \pi H^0 \pi = \mathcal{U}^{-1}(\Phi) \left( \int_{\mathbb{T}_x} \hat{\pi}(\theta) \hat{H}^0(\theta) \hat{\pi}(\theta) d\theta \right) \mathcal{U}, \]
\( h_o \in S^{-\infty}(\mathbb{Z}) \) be the symbol of the operator \( H_o := \pi H^0 \pi \), and \( h^*_o \in S^{-\infty}(\mathbb{Z}) \) be the symbol of its adjoint. Moreover, using (7.4) and then Proposition 7.1, we denote by \( \mathcal{R}_\perp \) the inverse of \( \pi^\perp H^0 \pi^\perp \) as operator acting in \( \pi^\perp \mathcal{H} \) and let \( r \in S^{-m}_1(\mathbb{Z}) \) be its symbol.
Using Proposition 8.1 and Proposition 6.2 and the fast decay of the modified Wannier function we can prove

\[
\langle \phi_{\gamma}^\epsilon, \Delta p^\epsilon((h_{\partial\partial}^\epsilon)^{\delta^\epsilon} + \delta^\epsilon h_{\partial\partial}^\epsilon) + \delta^\epsilon h_{\partial\partial}^\epsilon \delta^\epsilon + \eta^\epsilon h_{\partial\partial}^\epsilon \alpha^\epsilon \eta^\epsilon \phi_{\gamma 0}\rangle_{\chi} \\
= \langle \tau_{\gamma} \psi_0, Z \psi_0 \rangle_{\chi} + \epsilon O((\gamma)^{-m}),
\]

(8.38)

with

\[
Z := \Delta p((h_{\partial\partial}^\epsilon + \delta h_{\partial\partial}^\epsilon) + \delta h_{\partial\partial}^\epsilon + \eta h_{\partial\partial}^\epsilon \eta).
\]

(8.39)

Here we also used that \(\Lambda^\epsilon(\gamma, 0) = 1\) for any \(\gamma \in \Gamma\).

The operator \(Z\) has the form

\[
Z = H^0(Y^{-1/2} - 1) + (Y^{-1/2} - 1)H^0 + (Y^{-1/2} - 1)H^0(Y^{-1/2} - 1) \\
+ Y^{-1/2} \pi H^0 \pi R_\perp \pi H^0 \pi Y^{-1/2},
\]

(8.40)

and recall from (4.3) that

\[
Y := \pi H^0 \pi R_\perp \pi H^0 \pi.
\]

All the operators appearing in (8.40) have evidently \(\Gamma\)-periodic symbols and thus also \(Z\) and \(Y\). More precisely we have the following direct integral decomposition:

\[
Y = \pi H^0 \pi R_\perp \pi H^0 \pi \equiv H_\bullet R_\perp \pi H_\bullet = \int_{\mathbb{T}_*} d\theta \hat{Y}(\theta) \bigg|_{\Gamma},
\]

with

\[
\hat{Y}(\theta) := \hat{H}_\bullet(\theta) \hat{R}(\theta) \hat{H}_\bullet(\theta)^\ast
\]

and \(\hat{H}_\bullet(\theta)\) maps \(\hat{\pi}^{\perp}(\theta)\mathcal{F}_\theta\) into \(\hat{\pi}(\theta)\mathcal{F}_\theta\) (see (3.5) for the definition of \(\hat{\pi}(\theta)\)) and is defined by

\[
\hat{H}_\bullet(\theta) = \hat{\pi}(\theta) \left( \sum_{n \in \mathbb{N}} \lambda_n(\theta) \mid \hat{\phi}_n(\theta) \rangle \langle \hat{\phi}_n(\theta) \mid \right) \hat{\pi}(\theta)^\perp.
\]

Thus

\[
Y = \pi H^0 \pi R_\perp \pi H^0 \pi = \int_{\mathbb{T}_*} d\theta \hat{Y}(\theta) \bigg|_{\Gamma},
\]

with \(\hat{Y} \in C^\infty(\mathbb{T}_*; \mathbb{R}_+).\)
Taking into account the definitions and arguments in Section 3, we obtain that, for \( \theta \in \Sigma_b \),

\[
\hat{H}_\star(\theta) = |\hat{\phi}_0(\theta)\rangle \langle \hat{\phi}_0(\theta)| \left( \sum_{n \in \mathbb{N}} \lambda_n(\theta) |\hat{\phi}_n(\theta)\rangle \langle \hat{\phi}_n(\theta)| \right) (1 - |\hat{\phi}_0(\theta)\rangle \langle \hat{\phi}_0(\theta)|) = 0,
\]

so that we conclude that \( \hat{Y}(\theta) = 0 \) for \( \theta \in \Sigma_b \).

Now the operator \( Z \) in (8.39) is also \( \Gamma \)-decomposable and

\[
Z = \mathcal{U}_\Gamma^{-1} \left( \int_{\Gamma} d\theta \, \hat{Z}(\theta) \right) \mathcal{U}_\Gamma,
\]

where \( \hat{Z} \in C^\infty(\Gamma) \) satisfies \( \hat{Z}(\theta) = 0 \), for all \( \theta \in \Sigma_b \). Finally, we obtain that

\[
\hat{x}^\epsilon(\theta) - \hat{x}_0^\epsilon(\theta) = \sum_{\gamma \in \Gamma} e^{-i\langle \theta, \gamma \rangle} \langle \phi_{\gamma}^\epsilon, \mathcal{D}p^\epsilon (\mathcal{V}^\epsilon - h^\epsilon_x) \phi_0^\epsilon \rangle_{\mathcal{H}}
\]

\[
= \sum_{\gamma \in \Gamma} e^{-i\langle \theta, \gamma \rangle} \langle \tau_{\gamma} \psi_0, Z \psi_0 \rangle_{\mathcal{H}} + \mathcal{O}(\epsilon)
\]

\[
= \hat{Z}(\theta) + \mathcal{O}(\epsilon) = \mathcal{O}(\epsilon), \quad \text{for all } \theta \in \Sigma_b,
\]

which ends the proof of the proposition and of Theorem 2.2 (ii).

8.5. **Proof of corollaries 2.3 and 2.4.** First, an application of Proposition 8.12 and Proposition 4.5 with \( \eta = \epsilon \) shows that the spectrum of \( H^{\epsilon, \kappa} \) must have gaps of order \( \epsilon \) in the interval \([0, N\epsilon]\), provided the same is true for \( \mathcal{D}p^{\epsilon, \kappa}(\hat{x}^{\epsilon}) \).

Second, one has to perform the spectral analysis of \( \mathcal{D}p^{\epsilon, \kappa}(\hat{x}^{\epsilon}) \) applying the results in [4] and prove the existence of gaps of order \( \epsilon \) independent of \( \kappa \) in its spectrum restricted to the interval \([0, N\epsilon]\), provided \( \kappa \) and \( \epsilon \) are small enough.

- Assume \( B^\Gamma = 0 \). Then we notice that the function \( \hat{x}^{\epsilon}: \mathbb{T}_* \rightarrow \mathbb{R} \) has exactly the same properties as the function \( \lambda^{\epsilon}: \mathbb{T}_* \rightarrow \mathbb{R} \) defined in Definition 3.18 in [4] with the only difference that the estimate \( \|\partial^{\alpha} \lambda^{\epsilon}(\theta) - \partial^{\alpha} \lambda_0(\theta)\| \leq C_m \epsilon \) which was valid for all \( \theta \in \mathbb{T}_* \) (as stated in Proposition 4.1 in [4]) is now only valid on \( \Sigma_b \). Nevertheless, if we choose an upper bound \( \delta_0 \) for the cut-off parameter \( \delta > 0 \) in Subsection 4.3 (Paragraph ‘Cut-off functions near the minimum’) in [4] such that \( \{ \theta \in \mathbb{E}_* \mid |\theta| \leq \sqrt{2m^{-1}_T \delta_0} \} \subseteq \Sigma_b \), all the results of Section 4 of [4] remain true for our function \( \hat{x}^{\epsilon}: \mathbb{T}_* \rightarrow \mathbb{R} \) and Corollary 2.3 follows.
Assume $B^\Gamma \neq 0$. Then the situation is rather similar, but the function $\lambda_0: T_* \to \mathbb{R}$ may no longer be symmetric around its local minimum $\theta_0$. In this case, a third order term may appear in the Taylor expansion (4.1) in [4] and thus formula (4.3) in [4] becomes

$$
\lambda^\epsilon(\theta) - \lambda^\epsilon(\theta^\epsilon) = \sum_{1 \leq j,k \leq 2} a^\epsilon_{jk}(\theta_j - \theta_j^\epsilon)(\theta_k - \theta_k^\epsilon) + O(|\theta - \theta^\epsilon|^3).
$$

(8.41)

As a consequence, the exponent $\mu > 0$ relating the cut-off parameter $\delta > 0$ with the intensity of the magnetic field $\epsilon > 0$ through condition (4.20) in [4] may vary only in the interval $(2, 3)$ in $\mathbb{R}$. A “symmetric choice,” similar to the one in [4] is thus $\mu = 2.5$ and the only effect of this new choice is that in the second formula (4.28) in Proposition 4.5 in [4] we now have the estimate

$$
\|\mathcal{D}^\epsilon p^\epsilon,\kappa(\tau_{\delta,a})\| \leq C \epsilon^{1/5}.
$$

This ends the proof of Corollary 2.4.

References


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