Two weight inequalities for positive operators: doubling cubes

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Abstract. For the maximal operator $M$ on $\mathbb{R}^d$, and $1 < p, \rho < \infty$, there is a finite constant $D = D_{p, \rho}$ so that this holds. For all weights $w, \sigma$ on $\mathbb{R}^d$, the operator $M(\sigma \cdot)$ is bounded from $L^p(\sigma) \to L^p(w)$ if and only if the pair of weights $(w, \sigma)$ satisfy the two weight $A_p$ condition, and this testing inequality holds:

$$\int_Q M(\sigma 1_Q) \, dw \lesssim \sigma(Q),$$

for all cubes $Q$ for which there is a cube $P \supset Q$ satisfying $\sigma(P) \leq D\sigma(Q)$, and $\ell(P) \geq p\ell(Q)$. This was recently proved by Kangwei Li and Eric Sawyer. We give a short proof, which is easily seen to hold for several closely related operators.

1. Introduction

Our subject is the two weight inequalities for the maximal function, fractional integral transforms, and Poisson integrals. For the purposes of this section, we will focus on the maximal function. A weight $w$ is a non-negative Borel measure on $\mathbb{R}^d$, and given two weights $w, \sigma$ we say that $(w, \sigma) \in A_p$ if the constant

$$[w, \sigma]_p = \sup_Q \langle w \rangle_Q^{1/p} \langle \sigma \rangle_Q^{1/p'}, \quad p' = \frac{p}{p-1},$$

where here and throughout $\langle w \rangle_Q = |Q|^{-1} \int_Q w \, dx$. With this notation the maximal function is

$$Mf = \sup_{Q \text{ cube}} \langle f \rangle_Q 1_Q.$$

The classical two weight inequality for the maximal function due to Sawyer [15] is below. It shows that the inequality for the maximal function reduces to a testing inequality for indicators of cubes.

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Theorem 1.2 (Theorem B in [15]). For two weights \( (w, \sigma) \) we have the inequality
\[
\|M(\sigma f)\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}
\]
if and only if the testing inequality below holds:
\[
\sup_{Q : \sigma(Q) > 0} \sigma(Q)^{-1/p} \|1_Q M(\sigma 1_Q)\|_{L^p(w)} < \infty.
\]

Recent papers of Hytönen, Li and Sawyer [6], [12] began a study of a weaker class of testing inequalities in the two weight setting. (Their papers include interesting motivation and background.) They introduce four such conditions. The definition below is weaker than their weakest condition: test the maximal function on indicators of cubes \( Q \) which have some parent on which \( \sigma \) is doubling.

Definition 1.3. Given two weights \( (w, \sigma) \), and \( 1 < p, \rho, D < \infty \), we say that \( (w, \sigma) \) satisfy a \( (p, \rho, D) \) parent doubling testing condition if there is a positive finite constant \( \mathfrak{P} = \mathfrak{P}_{p, D} = \mathfrak{P}(w, \sigma, d, p, \rho, D) \) so that we have
\[
\|1_Q M(\sigma 1_Q)\|_{L^p(w)} \leq \mathfrak{P} \sigma(Q)^{1/p}, \tag{1.4}
\]
for every cube \( Q \) for which there is a second cube \( P \supset Q \), with \( \ell(P) \geq \rho \ell(Q) \), and \( \sigma(P) \leq D \sigma(Q) \).

Above \( \ell(Q) = |Q|^{1/d} \) is the side length of \( Q \). The case of \( \rho = p = 2 \) below is (just a little stronger than) the main result of Li and Sawyer [12].

Theorem 1.5. Let \( 1 < p, \rho < \infty \). There is a constant \( D = D_{d,p,\rho} \) so that for any pair of weights \( (\sigma, w) \) we have
\[
\|M(\sigma)\|_{L^p(\sigma) \to L^p(w)} \simeq [w, \sigma]_p + \mathfrak{P}_{p, D}.
\]

Our proof is short: any cube which is nearly maximal with respect to the \( A_p \) condition must also be doubling, hence satisfies the testing assumption. The cubes which are not maximal are trivial to control. The proof readily adapts to several closely related operators, as indicated in the concluding section.

2. Proof

The norm bound on \( M \) easily implies the two weight \( A_p \) condition on the weights, as well as the testing condition for all cubes, not just those with a doubling parent. The content of Theorem 1.5 is that the reverse implication holds.

Our theorem only claims that there is a sufficiently large doubling parameter \( D \) which can be used for all pairs of weights \( (w, \sigma) \). Below, we will consider values of \( 1 < \rho \leq 2 \). For integers \( n = 3, 4, \ldots \), and choices of \( n - 1 < \rho \leq n \), the argument proceeds by replacing the dyadic grids introduced below by \( n \)-ary grids. We omit the details.
By a dyadic grid we mean a collection $\mathcal{D}$ of cubes in $\mathbb{R}^d$ for which (a) if $P, Q \in \mathcal{D}$, then $P \cap Q$ is either empty, or $P$ or $Q$, and (b) for all integers $k$, the cubes $\{ Q \in \mathcal{D} : |Q| = 2^{kd} \}$ partition $\mathbb{R}^d$. Associated to the grid $\mathcal{D}$ is the maximal function

$$M_{\mathcal{D}} f = \sup_{Q \in \mathcal{D}} \langle f \rangle_Q 1_Q.$$ 

As is well known, there are a finite number of grids $\mathcal{D}_1, \ldots, \mathcal{D}_{3^d}$ for which

$$M f \lesssim \sup_{1 \leq j \leq 3^d} M_{\mathcal{D}_j} f. \tag{2.1}$$

See Lemma 2.6 of [5]. The implied constant above depends upon dimension. Set

$$D = 2^{d(p+1)/(p-1)}.$$ 

It suffices to show that under the two weight $A_p$ and $(p, 2, D)$ parent testing condition, for any dyadic grid $\mathcal{D}$, the maximal function $M_{\mathcal{D}}(\sigma \cdot)$ is bounded from $L^p(\sigma)$ to $L^p(w)$.

Sawyer’s Theorem 1.2 holds for $M_{\mathcal{D}}$. Namely, it suffices to show that for any cube $Q_0 \in \mathcal{D}$, we have

$$\int_{Q_0} M_{\mathcal{D}}(\sigma 1_{Q_0})^p \, dw \lesssim ([w, \sigma]_p^p + \mathcal{P}^p) \sigma(Q_0). \tag{2.2}$$

We are free to restrict the supremum defining the maximal function to the collection of cubes $Q = \{ Q \in \mathcal{D} : Q \subset Q_0 \}$ of cubes contained in $Q_0$.

Partition $Q$ into at most four subcollections using these definitions.

- **(Testing collection)** Let $T^*$ be the maximal elements $Q \in \mathcal{D}$ with $Q \subset Q_0$ so that the testing inequality (1.4) holds. Set $T_Q = \{ P \in Q : P \subset Q \}$, for $Q \in T^*$. And set $T = \bigcup_{Q \in T^*} T_Q$.

- **(The top)** Let $U = \{ Q \in Q \setminus T^* : 2^k \ell(Q) \geq \ell(Q_0) \}$. We choose $k$ large enough that $2^{dk}k^{-p} > 1$. These are the cubes which are close to the top cube $Q_0$.

- **(Small $A_p$ cubes)** Let $A$ be those cubes $Q \in Q \setminus (T \cup U)$ such that

$$\langle \sigma \rangle_Q^{1/p} \langle w \rangle_Q^{1/p} \leq \frac{[w, \sigma]_p}{\log \ell(Q_0)/\ell(Q)}. \tag{2.3}$$

That is, the local $A_p$ constant at $Q$ is very small.

- **(Remaining cubes)** Let $\mathcal{R} = Q \setminus (T \cup U \cup A)$.

We show that the maximal function over each collection satisfies the testing inequality (2.2). The ‘testing collection’ is very easy:

$$\int_{Q_0} \sup_{Q \in T} \langle \sigma \rangle_Q^p 1_Q \, dw \leq \sum_{Q \in T} \int_Q M(\sigma 1_Q)^p \, dw \leq \mathcal{P}_{2,D}^p \sum_{Q \in T} \sigma(Q) \leq \mathcal{P}^p \sigma(Q_0).$$
The ‘top collection’ $\mathcal{U}$ has at most $2^{1+d(k+1)}$ elements, and we just use the $A_p$ condition to see that

$$\int_{Q_0} \sup_{Q \in U} (\sigma)_{Q_0}^p 1_Q \, dw \leq \sum_{Q \in U} (\sigma)_{Q_0}^p \langle w \rangle_Q \leq [w, \sigma]^p \sum_{Q \in U} \sigma(Q) \lesssim [w, \sigma]^p \sigma(Q_0).$$

The implied constant depends on $k$, but that is a fixed integer.

The ‘small $A_p$ cubes’ are also trivially summed up, using the condition in (2.3):

$$\int_{Q_0} \sup_{Q \in A} (\sigma)_{Q_0}^p 1_Q \, dw \leq \sum_{Q \in A} (\sigma)_{Q_0}^p \langle w \rangle_Q \leq [w, \sigma]^p \sum_{Q \in A} \sigma(Q) \lesssim [w, \sigma]^p \sigma(Q_0).$$

The implied constant depends upon $k$ and $p > 1$.

Thus, the core of the argument is control of the ‘remaining cubes’, $R$. Indeed, we claim that this collection is empty, since a cube that has a large local $A_p$ product is also approximately doubling.

Suppose $R \neq \emptyset$. Thus, there is a cube $Q \subset Q_0$, which satisfies $\ell(Q) < 2^{-k} \ell(Q_0)$, fails (2.3), and no ancestor of $Q$ also contained inside of $Q_0$ has a doubling parent. The last condition is very strong.

Let $Q^{(0)}$ be the $D$-parent of $Q = Q^{(0)}$, and let $Q^{(t+1)} = (Q^{(t)})^{(1)}$. Define integer $m$ by $Q_0 = Q^{(m)}$. Note that since $Q \notin U$, we have a lower bound on $m$, namely $m > k$. For any integer $0 \leq t < m$, we necessarily have $\sigma(Q^{(t+1)}) > D\sigma(Q^{(t)})$, since $Q^{(t+1)}$ is a $\rho$-parent of $Q^{(t)}$. That is, $\sigma(Q_0) > D^m \sigma(Q)$. From this and that fact that $Q \notin A$, we see that $m$ cannot be very large, contradicting our lower bound on $m$:

$$[w, \sigma]^p \sigma(Q_0) \geq \langle w \rangle_{Q_0} > D^{m(p-1)} \left( \frac{\sigma(Q)}{[Q^{(m)}]} \right)^p \frac{\sigma(Q)}{[Q^{(m)}]} \geq \frac{D/2^{dp}}{m^{p-1}} \sigma(Q^{(m)})^{p-1} \langle w \rangle_Q \geq [w, \sigma]^p [D/2^{dp}]^{m(p-1)} m^{-p} \geq [w, \sigma]^p 2^{2dm} m^{-p}.$$ 

The constants are explicit, and the last inequality follows by choice of $D$. We see that $m < k$, which is a contradiction.

3. Complements

1. The conditions in Theorem 1.5 can be strengthened by adding the condition that the doubling cubes satisfy $\sigma(\partial D) = 0$. This is accomplished by selection of random grids. The discussion needed is given by Li and Sawyer [12], §2, and we omit the details. Similar comments apply to the extensions we mention below.

2. Theorem 1.5 has a straightforward extension to fractional maximal functions.
3. The method of proof easily extends to other operators which are well approximated by dyadic grids. One of these is the Poisson integral given by

$$Pf(x,t) = \int \frac{t}{(t^2 + |x-y|^2)^{(d+1)/2}} f(y) \, dy, \quad t > 0.$$  

Given weights $\sigma$ on $\mathbb{R}^d$ and $w$ on the upper half space $\mathbb{R}^{d+1}_+$, we remark that the role of cubes in $\mathbb{R}^{d+1}_+$ are played by Carleson cubes, namely $\tilde{Q} = Q \times [0, \ell(Q))$, for $Q \subset \mathbb{R}^d$. The definition of the two weight $A_p$ condition is then

$$[w,\sigma]_p = \sup_Q \frac{\langle w \rangle_{\tilde{Q}}^{1/p} \langle \sigma \rangle_{\tilde{Q}}^{1/p'}}{\sigma(Q)^{1/p}}.$$  

Using similar methods, one can prove this version of the Poisson two weight theorem of Sawyer [14]. We single out this statement since the Sawyer’s Poisson theorem is an important ingredient of the two weight inequality for the Hilbert transform [9], [7].

**Theorem 3.2.** Let $1 < p, p' < \infty$. There is a $D > 1$ so that this holds. Let $w$ be a weight on $\mathbb{R}^d$, $\sigma$ on $\mathbb{R}^d$. These conditions are necessary and sufficient for $P(\sigma \cdot)$ to map $L^p(\mathbb{R}^d, \sigma)$ to $L^{p'}(\mathbb{R}^{d+1}_+, w)$. There is a finite constant $\Psi$ so that

1. the $A_p$ condition (3.1) holds,
2. if $Q \subset \mathbb{R}^d$ is a cube for which $\sigma(\rho Q) < D\sigma(Q)$, then $\|1_Q P(\sigma 1_Q)\|_{L^p(\mathbb{R}^{d+1}_+, w)} \leq \Psi \sigma(Q)^{1/p},$
3. if $Q \subset \mathbb{R}^d$ is a cube for which $w(\rho \tilde{Q}) < Dw(\tilde{Q})$, then $\|1_Q P^*(w 1_\tilde{Q})\|_{L^{p'}(\mathbb{R}^d, \sigma)} \leq \Psi w(\tilde{Q})^{1/p'}.$

Let us briefly indicate the proof. For any dyadic grid $\mathcal{D}$, we can define

$$P_\mathcal{D} f = \sum_{Q \in \mathcal{D}} \langle f \rangle_Q 1_Q.$$  

One has $P_\mathcal{D} f \lesssim Pf$, but also an analog of (2.1) holds. That is, there are finitely many dyadic grids $\mathcal{D}_1, \ldots, \mathcal{D}_3^d$ for which

$$Pf \lesssim \sup_{1 \leq j \leq 3^d} P_{\mathcal{D}_j} f.$$  

It therefore remains to see that the three conditions in Theorem 3.2 imply that $P_\mathcal{D}$ is bounded, for any choice of grid.

There is a Sawyer type testing theorem for dyadic positive operators [10], so that it suffices to verify the testing inequality

$$\|1_Q P_\mathcal{D}(\sigma 1_Q)\|_{L^p(w)} \lesssim ([w,\sigma]_p + \Psi) \sigma(Q)^{1/p}.$$  

as well as the dual estimate. We have arrived at the point (2.2) in our proof of Theorem 1.5. The remaining steps easily extend to this setting.
4. One can also deduce a doubling parent testing type condition for the fractional integral operators
\[ T_\alpha f = \int f(x - y) \frac{dy}{|y|^{d+\alpha}}, \quad 0 < \alpha < 1. \]

**Theorem 3.3.** Let \( 1 < \rho < \infty \) and \( 1 < p \leq q < \infty \). There is a \( D > 1 \) so that this holds. Let \( (w, \sigma) \) be weights on \( \mathbb{R}^d \). These conditions are necessary and sufficient for \( P(\sigma) \) to map \( L^p(\mathbb{R}^d, \sigma) \) to \( L^q(\mathbb{R}^d, w) \). There is a finite constant \( \mathfrak{P} \) so that

1. the pair of weights satisfy the \( A_{p,q,\alpha} \) condition \( \sup_Q w(Q)^{1/q} \sigma(Q)^{1/p'}/|Q|^\alpha \),
2. if \( Q \subset \mathbb{R}^d \) is a cube for which \( \sigma(\rho Q) < D \sigma(Q) \), then \( \|1_Q T_\alpha(\sigma 1_Q)\|_{L^q(w)} \leq \mathfrak{P} \sigma(Q)^{1/p'} \),
3. if \( Q \subset \mathbb{R}^d \) is a cube for which \( w(\rho Q) < D w(Q) \), then \( \|1_Q T_\alpha(w 1_Q)\|_{L^{q'}(\sigma)} \leq \mathfrak{P} w(Q)^{1/q'} \).

There is a corresponding characterization of the weak type inequality. The sketch of Theorem 3.2 applies to the theorem above.

5. It is an interesting question to gain information about the optimal choice of \( D = D_{\rho,p} \) in Theorem 1.5. We have not sought to do so, and comment briefly on the case of \( \rho = p = 2 \). It is clear that \( D \) cannot be very small, because then the allowed cubes on which one tests the norm of the maximal function are just too few, or subcritical for the pair of weights.

For instance, one knows that for weights \( w \in A_2 \) that one has
\[ \|M\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}. \]
This estimate is sharp in the power of the \( A_2 \) constant, as is shown by considering the weight \( w(x) = |x|^{d-\epsilon} \) for \( 0 < \epsilon < 1/2 \). One can calculate that \([w]_{A_2} \simeq \epsilon^{-1}\), with the cubes that demonstrate this being those centered at the origin. Note that with cube \( Q \) centered at the origin, one has
\[ w(Q) \simeq (\ell(Q))^{2d-\epsilon}. \]
It follows that the best possible choice of \( D = D_{2,2} \) would have to be of the order of \( 2^{2d} \). We have shown that \( D = 2^{2d} \) is sufficient.

6. For the values of \( 1 < \rho < 2 \), we are providing a very poor estimate of \( D_{\rho,p} \). Indeed, one would suspect that \( D_{\rho,p} \to 1 \) as \( \rho \downarrow 1 \). To show this, one would seem to need an improved notion of a shifted grids. The appendix of [1] gives one suggestion. Similar sorts of questions have been addressed in [3], [2].

7. It is of interest to extend the results of this paper to non-positive operators. One easy remark is this. Let \( D \) be a dyadic grid in \( \mathbb{R}^d \), and let \( \{\Delta_Q : Q \in D\} \) be the associated martingale differences. Define a martingale transform by
\[ Tf = \sum_{Q \in D} \epsilon_Q \Delta_Q f, \quad \epsilon_Q \in \{\pm 1\}. \]
For the two weight inequality, one has the result of Nazarov–Treil–Volberg [13] in the $L^2$ case. This leads to the following sufficient conditions for a two weight inequality.

**Theorem 3.4.** Let $(w, \sigma)$ be weights on $\mathbb{R}^d$ which satisfy the $A_2$ condition (1.1). There is a constant $D > 1$ so that these two conditions are sufficient conditions for $T(\sigma \cdot)$ to map $L^2(\sigma)$ to $L^2(w)$: for some finite constant $\mathfrak{P}$,

1. for all cubes $Q \in \mathcal{D}$ with $\sigma(Q^{(1)}) < D\sigma(Q)$, there holds $\|1_Q T(\sigma 1_Q)\|_{L^2(w)} \leq \mathfrak{P} \sigma(Q)^{1/2}$,

2. the same condition above holds, with the roles of $\sigma$ and $w$ reversed.

Above, $Q^{(1)}$ is the dyadic parent of $Q$.

We state this in the case of $p = 2$, as the $L^p$-case is much more complicated, see Vuorinen [16]. The weak-type inequality for maximal truncations of martingale transforms does admit a testing characterization. See Theorem 4.3 in [4]. One can consult [11] for information about the continuous case.

**8.** Certain kinds of $g$-functions have a two weight characterization [8]. That theorem can probably be relaxed to the current setting. More involved would be the weak-type estimate for maximal truncations of singular integrals, characterized in [4].

**9.** Potentially more interesting is relaxing the testing conditions in the two weight inequality for the Hilbert transform [7], [9]. It seems very likely that such a result is true.

**References**


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