Joint rotational and stretching multifractal spectra of mappings with integrable distortion

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Abstract. We establish bounds for both the stretching and the rotational multifractal spectra of planar homeomorphic mappings with $p$-integrable distortion. Moreover, we show that these bounds are sharp by presenting extremal examples of homeomorphic mappings with $p$-integrable distortion that attain the optimal stretching and rotation simultaneously, which additionally leads to the sharp bound for the joint rotational and stretching multifractal spectra. Finally, we will use the stretching multifractal spectra to obtain the sharp area contraction for homeomorphic mappings with $p$-integrable distortion.

1. Introduction

We consider stretching and rotational properties of homeomorphic mappings of finite distortion, and for the purpose of this paper all mappings of finite distortions are assumed to be homeomorphic.

Let $f : \mathbb{C} \to \mathbb{C}$ be a mapping of finite distortion, for which $K_f(z) \in L^p_{\text{loc}}(\mathbb{C})$. Then it is well known that its pointwise stretching is bounded by

$$ |f(z) - f(x)| \geq e^{-c(1/|z-x|)^{2/p}}, $$

where $c = c_{f,p,z}$ and $|z - x| > 0$ is small, see, for example, [10] and [3]. Moreover, recently in [9] the pointwise rotation of such mappings was bounded by

$$ |\arg(f(z + h) - f(z))| \leq c \left( \frac{1}{|h|} \right)^{2/p}, $$

where $c = c_{f,p,z}$ and $|h| > 0$ is small. Furthermore, both of these bounds have been shown to be sharp in the sense that the exponent $2/p$ can not be made any smaller.

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The sharp pointwise bounds (1.1) and (1.2) provide a starting point for studying the size of sets in which a mapping with \( p \)-integrable distortion can attain some slightly weaker stretching, rotation or both of them simultaneously. Identifying the maximal size of these sets amounts to finding the stretching, rotational or joint multifractal spectra of these mappings. The sharp bounds for the multifractal spectra, which we prove in this paper, describe accurately many geometrical properties of these mappings. For example, we use the stretching multifractal spectra to prove the sharp bound for the area contraction of mappings with \( p \)-integrable distortion, which improves the result given in [3]. Moreover, we believe that the multifractal spectra should have many other applications in the field, and that the stretching multifractal spectra can be extended to the higher dimensions using similar methods as in the planar case.

My interest in these questions has been partly inspired by [2], where the complete description of the joint multifractal spectra for quasiconformal mappings was given. However, the methods used in [2] rely heavily on the quasiconformality, and thus do not readily extend for more general families of mappings. Hence we have to use a different approach and apply the modulus of path families as in the pointwise case, see [8] and [9], when studying mappings with \( p \)-integrable distortion. In order for this approach to work we will have to overcome some difficulties related to finding suitable path families. Our first result gives the sharp bound for the stretching multifractal spectra of these mappings.

**Theorem 1.1.** Let \( p \geq 1, b > 0 \) and \( s \in (0, 2) \) be given and assume that \( f: \mathbb{C} \to \mathbb{C} \) is a homeomorphic mapping of finite distortion for which \( K_f(z) \in L^p_{\text{loc}}(\mathbb{C}) \). Let \( A \subset \mathbb{C} \) be the set of points \( z \in \mathbb{C} \) for which there exists a sequence of complex numbers \( \lambda_{z,n} \), where the moduli \( |\lambda_{z,n}| \to 0 \) form a decreasing sequence, satisfying

\[
|f(z + \lambda_{z,n}) - f(z)| \leq e^{-b(1/|\lambda_{z,n}|)^{(2-s)/p}},
\]

for every \( n \). Then \( \dim(A) \leq s \).

Our second result gives the sharp bound for the rotational multifractal spectra.

**Theorem 1.2.** Let \( p \geq 1, b > 0 \) and \( s \in (0, 2) \) be given, and assume that \( f: \mathbb{C} \to \mathbb{C} \) is a homeomorphic mapping of finite distortion for which \( K_f(z) \in L^p_{\text{loc}}(\mathbb{C}) \). Let \( A \subset \mathbb{C} \) be the set of points \( z \in \mathbb{C} \) for which there exist a branch of the argument and a sequence of complex numbers \( \lambda_{z,n} \), where the moduli \( |\lambda_{z,n}| \to 0 \) form a decreasing sequence, satisfying

\[
|\arg(f(z + \lambda_{z,n}) - f(z))| \geq b\left(\frac{1}{|\lambda_{z,n}|}\right)^{(2-s)/p},
\]

for every \( n \). Then \( \dim(A) \leq s \).

Note that the choice of the branch of the argument in Theorem 1.2 plays very little role, since any change to the branch of the argument in the inequality (1.4) changes the left-hand side with a fixed constant, which is insignificant as the radii \( |\lambda_{z,n}| \) converge to zero.
Remark 1.3. We would like to point out that in Theorems 1.1 and 1.2 we could instead of fixing the constant $b$, appearing in the conditions (1.3) and (1.4), assume that the constants $b = b(z) > 0$ can depend on the points $z \in A$ and still obtain the same result.

Let us briefly verify this in the case of stretching, the rotational case can be checked in a similar manner.

Fix an arbitrary $f$ and denote for a moment by $A_b$ the set of points that satisfy the condition (1.3) with a fixed constant $b > 0$. Then let the set $A$ consist of all the points $z \in C$ for which the condition (1.3) holds with some constant $b(z) > 0$, and note that

$$A \subset \bigcup_{m=1}^{\infty} A_{1/m}.$$ 

Hence, if we can prove that $\dim(A_{1/m}) \leq s$ for every $m$, then it follows that $\dim(A) \leq s$. Thus Theorem 1.1 implies also this more general looking result.

Furthermore, we will show that Theorems 1.1 and 1.2 are sharp, in the sense of the Hausdorff dimension, by constructing the following extremal mappings.

Theorem 1.4. Let $p \geq 1$ and $s \in (0, 2)$ be given. Then we can find a homeomorphism $f : C \to C$ with $p$-integrable distortion and a set $A \subset C$, for which $\dim(A) = s$, such that for every point $z \in A$ there exist a branch of the argument and a sequence $\lambda_{z,n}$, where $|\lambda_{z,n}| \to 0$, satisfying

$$|f(z + \lambda_{z,n}) - f(z)| \leq e^{-\left(\frac{1}{|\lambda_{z,n}|}\right)^{(2-s)/p}},$$

and

$$|\arg(f(z + \lambda_{z,n}) - f(z))| \geq \left(\frac{1}{|\lambda_{z,n}|}\right)^{(2-s)/p},$$

for every $n$.

Note that the choice of the branch of the argument in Theorem 1.4 plays again very little role.

In addition to the sharpness of Theorems 1.1 and 1.2, Theorem 1.4 also shows that the sharp stretching and rotation can happen simultaneously in a set of points that has the right Hausdorff dimension. This proves that the joint rotational and stretching multifractal spectra is the same, in the sense of the Hausdorff dimension, as the stretching or rotational multifractal spectra.

It remains an open question if the set $A$ in Theorems 1.1 and 1.2 can have positive or $\sigma$-finite Hausdorff $s$-measure. The construction by Clop and Herron, see Example 4.4 in [3], using generalized Cantor dust, hints that this might be possible. Additionally, one could even ask if the sets $A$ could have positive measure under some more general gauge functions.

Finally, we apply Theorem 1.1 to study the area contraction in the spirit of Clop and Herron, see [3]. Their idea was to use the modulus of continuity result (1.1) to estimate the compression of small balls under mappings with $p$-integrable distortion. Using this method, they proved that if $f : C \to C$ is a
mapping with $p$-integrable distortion and $A \subset \mathbb{C}$ satisfies $H^s(A) > 0$, then the image set satisfies $H^b(f(A)) > 0$, where the gauge function is defined by

$$h(t) = \left(\frac{1}{\log(1/t)}\right)^{ps/2}.$$  

Moreover, they constructed examples of mappings $f: \mathbb{C} \to \mathbb{C}$ with $p$-integrable distortion that can map a set $A \subset \mathbb{C}$, with $H^s(A) > 0$ and $s \in (0, 2)$, to a set which satisfies $H^b(f(A)) = 0$, where

$$h(t) = \left(\frac{1}{\log(1/t)}\right)^{ps/(2-s)}.$$  

As there was a gap left between the gauge functions $h(t)$ and $\overline{h}(t)$, see the exponents in (1.5) and (1.6), Clop and Herron asked if the bound (1.5) can be improved. We give a positive answer to this question by using Theorem 1.1, instead of the modulus of continuity result (1.1) which was used in [3], to get a better control for the distortion of small balls and obtain

**Theorem 1.5.** Let $f: \mathbb{C} \to \mathbb{C}$ be a homeomorphic mapping of finite distortion such that $K_f(z) \in L^p_{\text{loc}}(\mathbb{C})$, where $p \geq 1$. Fix $s \in (0, 2)$ and define the gauge function

$$h(t) = \left(\frac{1}{\log(1/t)}\right)^{ps/(2-s)},$$

where $\epsilon > 0$ can be chosen arbitrary small. Then every $A \subset \mathbb{C}$ with $H^b(f(A)) = 0$ satisfies $H^s(A) = 0$.

Furthermore, examples constructed by Clop and Herron that compress as gauge functions (1.6), together with Theorem 1.5, show that the gauge function

$$h(t) = \left(\frac{1}{\log(1/t)}\right)^{ps/(2-s)}$$

is indeed the critical one when measuring the area compression.

If we would choose $s = 2$ in Theorem 1.5, the gauge function in (1.7) would not make sense as the exponent would approach infinity when $\epsilon \to 0$. In this case we are not sure how the correct gauge function $h$ should look like.

### 2. Prerequisites

Let $\Omega \subset \mathbb{C}$ be a domain and $f: \Omega \to \mathbb{C}$ a sense-preserving homeomorphism. We say that $f$ has finite distortion if the following conditions hold:

- $f \in W^{1,1}_{\text{loc}}(\Omega)$,
- $J_f(z) \in L^1_{\text{loc}}(\Omega)$,
- $|Df(z)|^2 \leq J_f(z)K(z)$ almost everywhere in $\Omega$,
for a measurable function $K(z) \geq 1$, which is finite almost everywhere. The smallest such function is denoted by $K_f(z)$ and called the distortion of $f$. Here $Df(z)$ denotes the differential matrix of $f$ at the point $z$, and the norm $|Df(z)|$ is defined by

$$|Df(z)| = \max\{|Df(z)e| : e \in \mathbb{C}, |e| = 1\},$$

whereas $J_f(z)$ is the Jacobian of the mapping $f$ at the point $z$. Such a mapping is said to have a $p$-integrable distortion, where $p \geq 1$, if

$$K_f(z) \in L^p_{\text{loc}}(\Omega).$$

For a detailed exposition of mappings of finite distortion see, for example, [1] or [6].

Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism, let $g : (0, 1] \to [1, \infty)$ be a function such that $g(r) \to \infty$ as $r \to 0$, and fix an arbitrary branch of the argument. We say that the rotation of $f$ at the point $z_0 \in \mathbb{C}$ grows at the rate of the function $g$ if

$$\limsup_{r \to 0} \sup_{\theta \in [0, 2\pi]} \frac{\arg(f(z_0 + re^{i\theta}) - f(z_0))}{g(r)} = c, \tag{2.1}$$

for some constant $c > 0$. Note that the limit (2.1) does not depend on the choice of the branch of the argument.

In this light, Theorem 1.2 states that if $f$ is a mapping with $p$-integrable distortion and $A$ is the set of points in which the rotation of $f$ grows at least with the rate

$$g(r) = r^{-(2-s)/p}, \tag{2.2}$$

then $\dim(A) \leq s$. Moreover, the sharpness result Theorem 1.4 states that we can find a mapping $f$, with $p$-integrable distortion, and a set $A \subset \mathbb{C}$, with $\dim(A) = s$, such that the rotation of $f$ grows at least with the rate (2.2) at every point $z \in A$.

Let us then describe how we arrive at the definition (2.1) for the growth rate. When we study the pointwise rotation of a mapping $f$ at a point $z_0 \in \mathbb{C}$, we are interested in the change of the argument of $f(z_0 + te^{i\theta}) - f(z_0)$ as the parameter $t$ goes from 1 to $r > 0$, which we can write as

$$|\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|.$$ 

This can also be understood as the winding of the path $f([z_0 + re^{i\theta}, z_0 + e^{i\theta}])$ around the point $f(z_0)$. As we are interested in the maximal change of the argument, over an arbitrary direction $\theta$, we must study the supremum

$$\sup_{\theta \in [0, 2\pi]} |\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|. \tag{2.3}$$

Finally, we can study the growth of (2.3) at the limit $r \to 0$ and say that the rotation of $f$ at the point $z_0$ grows at the rate of a function $g : (0, 1] \to [1, \infty)$, which satisfies $g(r) \to \infty$ as $r \to 0$, if

$$\limsup_{r \to 0} \sup_{\theta \in [0, 2\pi]} \frac{|\arg(f(z_0 + re^{i\theta}) - f(z_0)) - \arg(f(z_0 + e^{i\theta}) - f(z_0))|}{g(r)} = c \tag{2.4}$$
for some constant $c > 0$. Note that the term $\arg(f(z_0 + e^{i\theta}) - f(z_0))$ and the choice of the branch of the argument play no role at the limit (2.4), and hence we arrive at the definition (2.1).

In the proofs of Theorems 1.1 and 1.2, we will rely on the modulus of path families. We will give here the main definitions; for a closer look on the topic we recommend, for example, [12].

We call an image of a continuous function $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ is an interval, a path and denote both the function and its image by $\gamma$. Let $\Gamma$ be a family of paths. We say that a Borel-measurable function $\rho : \mathbb{C} \to [0, \infty)$ is admissible with respect to $\Gamma$ if

$$\int_{\gamma} \rho(z) \, |dz| \geq 1,$$

for any locally rectifiable path $\gamma \in \Gamma$. We denote the modulus of a path family $\Gamma$ by $M(\Gamma)$ and define it by

$$M(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z) \, dz.$$

We will also need a weighted version of (2.6), where the weight function $\omega : \mathbb{C} \to [0, \infty)$ is measurable and locally integrable, which we define by

$$M_\omega(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z) \omega(z) \, dz.$$

We say that a homeomorphism $f : \Omega \to \mathbb{C}$ satisfies the Lusin $(N)$ condition if for each $E \subset \Omega$ holds

$$|E| = 0 \Rightarrow |f(E)| = 0,$$

where $|E|$ denotes the Lebesgue measure of the set $E$.

We use $c$ as a generic constant whose value might change even in the middle of inequalities, and we use $c_a$ if the constant depends on a parameter $a$. The boundary of a given set $A$ is denoted by $\partial A$, the unit disc by $\mathbb{D}$, the radius of a given ball $B$ by $r(B)$, and for any ball $B$ and a constant $c$ we denote $cB(a,r) = B(a,cr)$.

3. Background for proofs

The proofs of Theorems 1.1 and 1.2 rely on the following modulus inequality.

**Lemma 3.1.** Let $\Omega$ be a domain and let $f : \Omega \to \mathbb{C}$ be a homeomorphic mapping of finite distortion for which $K_f(z) \in L^1_{\text{loc}}(\Omega)$. Then, given a family $\Gamma$ of paths $\gamma \subset \Omega$, we have

$$M(f(\Gamma)) \leq M_{K_f}(\Gamma).$$

Note that we make no assumptions on the Lusin $(N)$ condition in Lemma 3.1. For the proof of this lemma, see [9], where this modulus inequality was used to determine the optimal pointwise rotation for mappings with integrable distortion.
We will use the modulus inequality (3.1) with a path family $\Gamma$ consisting of all the paths connecting suitable compact sets $E_j$, which are line segments, to suitable compact sets $F_j$, which are unions of a line segment and a circle. We will define these sets rigorously later, but for a sketch of one of the pairs $E_j$ and $F_j$, see Figure 1.

![Figure 1](image)

Figure 1. The sets $E_j$ and $F_j$.

For these path families $\Gamma$ to be helpful in the proofs of Theorems 1.1 and 1.2, we will have to find many disjoint circle like sets $F_j$, which are centered at points $z_j \in A$. Furthermore, in the case of Theorem 1.1 we must also show that the paths $f(E_j)$ get stretched enough under the mapping $f$, while in the case of Theorem 1.2 we must show that it is enough to study the rotation of the paths $f(E_j)$ around the points $f(z_j)$. To this end we formulate the following auxiliary results, where the first two concern the paths $f(E_j)$ and the third the number of the sets $F_j$.

First we prove that the assumptions of Theorem 1.1 imply that we can find for every point $z \in A$ a contracting sequence of line segments that satisfy strong stretching conditions. Moreover, we show that these line segments can be chosen to be of form $[z + \Lambda_{z,n}, z + C_0 \Lambda_{z,n}]$, where $|\Lambda_{z,n}| = 1/C_0^{m_{z,n}}$ for some positive integers $m_{z,n}$ and a fixed constant $C_0 > 1$ that does not depend on the point $z$.

Lemma 3.2. Fix parameters $a > 0$ and $b > 0$, and let $f$ be a homeomorphism. Assume that we can find a point $z \in C$ with a sequence of complex numbers $\lambda_n$, such that the moduli $|\lambda_n| \to 0$ form a decreasing sequence, which satisfies

$$|f(z + \lambda_n) - f(z)| \leq e^{-b(1/|\lambda_n|)^a}$$

for every $n$. Then we can find a sequence of complex numbers $\Lambda_n$, whose moduli satisfy $|\Lambda_n| = 1/C_0^{m_n}$ for some fixed constant $C_0 > 1$ and some increasing sequence of positive integers $m_n$, such that from every line segment $[z + \Lambda_n, z + C_0 \Lambda_n]$ we can find points $x$ and $y$ satisfying

$$\frac{|f(x) - f(z)|}{|f(y) - f(z)|} \leq e^{-\frac{b}{2}(1/|\Lambda_n|)^a}.$$
Proof. Without loss of generality we can assume that \( z = 0 \) and \( f(z) = 0 \). Fix
\[
C_0 = 4^{1/a},
\]
and note that \( C_0 > 1 \). Let us then assume that there does not exist such a sequence \( \Lambda_n \) and derive a contradiction.

Thus, we claim for a moment that there exists \( m \in \mathbb{N} \) such that we can not find points \( x, y \) satisfying the inequality (3.3) from any segment \([e^{i\theta}/C_0^n, e^{i\theta}/C_0^{n-1}]\), where \( n > m \) and \( \theta \in [0, 2\pi) \). Denote \( l = \min_{\theta \in [0, 2\pi)} |f(e^{i\theta}/C_0^n)| \) and write \( |f(\lambda_n)| \), where \( |\lambda_n| \in [1/C_0^{M+1}, 1/C_0^M] \) with \( M > m \), in the form
\[
|f(\lambda_n)| = \frac{|f(\lambda_n)|}{|f(\lambda_n)|} \cdot \frac{|f(\lambda_n)|}{|f(\lambda_n)|} \cdot \ldots \cdot \frac{|f(\lambda_n)|}{|f(\lambda_n)|} \cdot \frac{|f(\lambda_n)|}{|f(\lambda_n)|}.
\]
Estimate this using the assumption that given any segment \([e^{i\theta}/C_0^n, e^{i\theta}/C_0^{n-1}]\), where \( n > m \) and \( \theta \in [0, 2\pi) \), we can not find points \( x, y \) such that the inequality (3.3) holds, to obtain
\[
|f(\lambda_n)| \geq l \exp \left( -\frac{b}{16} \sum_{j=0}^{M+1} C_0^{aj} \right) = l \exp \left( -\frac{b}{16} \sum_{j=0}^{M+1} 4^j \right) \geq l e^{-b4^M/3}.
\]
On the other hand, we obtain from (3.2) that
\[
|f(\lambda_n)| \leq e^{-b(1/|\lambda_n|)^a} \leq e^{-bC_0^{Ma}} = e^{-b4^M}.
\]
When \( |\lambda_n| \to 0 \) we see that \( M \to \infty \), and thus the above estimates for \( |f(\lambda_n)| \) yield a contradiction. Hence the assumption is false and the claim holds.

Let us then prove in an analogous manner that the assumptions of Theorem 1.2 imply the existence of sequences of line segments that satisfy strong rotational conditions and have the same form as in the stretching case.

Lemma 3.3. Fix parameters \( a > 0 \) and \( b > 0 \), and let \( f \) be a homeomorphism. Assume that we can find a point \( z \in \mathbb{C} \) with a sequence of complex numbers \( \lambda_n \), for which the moduli \( |\lambda_n| \to 0 \) form a decreasing sequence, and a branch of the argument which satisfy
\[
|\arg(f(z + \lambda_n) - f(z))| \geq b \left( \frac{1}{|\lambda_n|} \right)^a
\]
for every \( n \). Then we can find a sequence of complex numbers \( \Lambda_n \), whose moduli satisfy \( |\Lambda_n| = 1/C_0^{m_n} \) for some fixed constant \( C_0 > 1 \) and some increasing sequence of positive integers \( m_n \), such that from every line segment
\[
[z + \Lambda_n, z + C_0\Lambda_n]
\]
we can find points \( x \) and \( y \) satisfying
\[
|\arg(f(z + x) - f(z)) - \arg(f(z + y) - f(z))| \geq b \left( \frac{1}{|\lambda_n|} \right)^a.
\]
Proof. We can again assume without loss of generality that $z = 0$ and $f(0) = 0$, and fix the constant $C_0 = 4^{1/\alpha} > 1$ as before. Then assume that we can not find such a sequence $\Lambda_n$, and derive a contradiction.

So, let $m \in \mathbb{N}$ be such that we can not find points $x$ and $y$ satisfying (3.6) from any segment $[e^{i\theta}/C_0^m, e^{i\theta}/C_0^{m-1}]$, where $n > m$ and $\theta \in [0, 2\pi)$. Denote

$$\alpha = \max_{\theta \in [0, 2\pi)} |\arg(f(e^{i\theta}/C_0^m))|,$$

and estimate $|\arg(f(\lambda_n))|$, where $|\lambda_n| \in [1/C_0^{M+1}, 1/C_0^M]$ and $M > m$, by

$$|\arg(f(\lambda_n))| \leq \alpha + \left| \arg\left(f\left(\frac{\lambda_n}{|\lambda_n| C_0^{n+1}}\right)\right) - \arg\left(f\left(\frac{\lambda_n}{|\lambda_n| C_0^n}\right)\right) \right| + \cdots$$

$$+ \left| \arg\left(f(\lambda_n)\right) - \arg\left(f\left(\frac{\lambda_n}{|\lambda_n| C_0^0}\right)\right) \right|.$$

Then estimate this using the assumption that for any segment $[e^{i\theta}/C_0^n, e^{i\theta}/C_0^{n-1}]$, where $n > m$ and $\theta \in [0, 2\pi)$, we can not find points $x$ and $y$ such that (3.6) would hold, to obtain

$$|\arg(f(\lambda_n))| \leq \alpha + \frac{b}{16} \sum_{j=0}^{M+1} C_0^{n+1} = \alpha + \frac{b}{16} \sum_{j=0}^{M+1} 4^j \leq \alpha + \frac{b}{3} 4^M.$$

On the other hand, from (3.5) we obtain that

$$|\arg(f(\lambda_n))| \geq \frac{b}{16} \left(\frac{1}{|\lambda_n|}\right)^a \geq b C_0^a M = b 4^M.$$

When $|\lambda_n| \to 0$ we see that $M \to \infty$, and thus the above estimates yield a contradiction. Hence the assumption is wrong and the claim holds.

Lemmas 3.2 and 3.3 show that when proving Theorems 1.1 and 1.2 we can instead of the sequences $\lambda_{z,n}$ consider the sequences $\Lambda_{z,n}$, whose moduli satisfy $|\Lambda_{z,n}| = 1/C_0^{m_{z,n}}$ with the choice $C_0 = 4^{p/(2-s)}$, and study the line segments $[z + \Lambda_{z,n}, z + C_0\Lambda_{z,n}]$ that satisfy either the stretching condition (3.3), or the rotation condition (3.6). The sets $E_j$ in Figure 1 will consist of these line segments.

Next we prove a lemma which ensures that we can find arbitrary small radii

$$R_{n_0} = 1/C_0^{a_0}$$

such that there exist many points $z_j \in A$ with the line segments of the form

$$[z_j + \Lambda_{z_j}, z_j + C_0\Lambda_{z_j}],$$

where $|\Lambda_{z_j}| = R_{n_0}$, that satisfy either the stretching condition (3.3) or the rotational condition (3.6). Moreover, we show that we can choose these points $z_j$ so that the circle like sets $F_j$, centered at the points $z_j$ with a fixed radius, are disjoint, and that each of them encircles the corresponding line segment $[z_j + \Lambda_{z_j}, z_j + C_0\Lambda_{z_j}]$. Multifractal spectra for stretch and rotation
Lemma 3.4. Fix any $\delta \in (0, 1)$, $s \in (0, 2)$ and $\alpha > 0$, and let $A \subset \mathbb{C}$. Assume that $\dim(A) = s$ and associate to every point $z \in A$ a decreasing sequence of radii $\{\delta^{k z, n}\}_{n=1}^{\infty}$, where $k z, n$ is a sequence of positive integers. Then, for any given $s_0 < s$, there exist radii $\delta^k$, which we can choose as small as we wish, such that we can find $\left\lfloor \frac{1}{\alpha \delta^k} \right\rfloor$ disjoint balls $B(z_j, \alpha \delta^k)$, where $z_j \in A$ and $\delta^k \in \{\delta^{k z, n}\}_{n=1}^{\infty}$ for every $j$.

Proof. Let us again assume that the claim does not hold and derive a contradiction. So, assume for a moment that there exists $s_1 < s$ such that for any radius $\delta^k$, where $k > k_1$, there exist less than $\left\lfloor \frac{1}{\alpha \delta^k} \right\rfloor$ disjoint balls $B(z_j, \alpha \delta^k)$ satisfying the conditions $z_j \in A$ and $\delta^k \in \{\delta^{k z, n}\}_{n=1}^{\infty}$ for every $j$.

Choose an arbitrary integer $k_1$ and denote by $A_{k_1}$, where $k > k_1$, the set of points $z \in A$ such that $\delta^k \in \{\delta^{k z, n}\}_{n=1}^{\infty}$. Note that the set $A_{k_1}$ might be empty, in which case we move on to the next integer. When the set $A_{k_1}$ is non-empty we fix for every $z \in A_{k_1}$ the ball $B(z_j, \alpha \delta^k)$, and use Vitali’s covering theorem to select points $z_j \in A_{k_1}$, such that the balls $B(z_j, \alpha \delta^k)$ are disjoint and

$$A_{k_1} \subset \bigcup_{j \in J_{k_1}} B(z_j, 5 \alpha \delta^k).$$

Moreover, since the balls $B(z_j, \alpha \delta^k)$, with $z_j \in A$ and $\delta^k \in \{\delta^{k z, n}\}_{n=1}^{\infty}$ for every $j$, are disjoint, our assumption says that there are less than $\left\lfloor \frac{1}{\alpha \delta^k} \right\rfloor$ balls $B_{j,k} = B(z_j, 5 \alpha \delta^k)$ in the cover

$$\bigcup_{j \in J_{k_1}} B_{j,k}.$$

Given any integer $k_1$, it is easy to see that

$$A \subset \bigcup_{k > k_1} A_k,$$

and hence, using (3.7), we obtain

$$A \subset \bigcup_{k > k_1} \bigcup_{j \in J_k} B_{j,k}.$$

Thus we can estimate the $s_2$-dimensional Hausdorff measure, where $s_1 < s_2 < s$, of the set $A$ by calculating

$$\sum_{k > k_1} \sum_{j \in J_k} \dim(B_{j,k})^s \leq 10^{s_2} s_2 \sum_{k > k_1} \sum_{j \in J_k} \frac{s_2}{s_2} \left(\frac{1}{\alpha \delta^k}\right)^{s_1}$$

$$\leq 10^{s_2} s_2 \sum_{k = 1}^{\infty} \frac{\delta^{s_2}}{\delta^{s_1}} \left(\frac{\delta^{s_2}}{\delta^{s_1}}\right)^k < M < \infty.$$

Since the radii of the balls $B_{j,k}$ can be made arbitrary small by choosing big $k_1$, we deduce that $H^{s_2}(A) < \infty$. But this is a contradiction since $\dim(A) = s$, where $s > s_2$. Thus the assumption does not hold and the claim follows. \qed

Armed with these auxiliary lemmata and the modulus inequality, we move on to prove Theorems 1.1 and 1.2.
4. Proof of Theorem 1.1

Fix parameters \( p \geq 1, b > 0 \) and \( s \in (0, 2) \), and let \( f : \mathbb{C} \to \mathbb{C} \) be a mapping of finite distortion such that \( K_f(z) \in L^p_{\text{loc}}(\mathbb{C}) \). Since we are interested in the Hausdorff dimension of the set \( A \) we can without loss of generality assume that \( A \subset \mathbb{D} \). Using Lemma 3.2, with the choice \( a = (2-s)/p \), we can find for every point \( z \in A \) a sequence of complex numbers \( \Lambda_{z,n} \), whose moduli form a decreasing sequence

\[
\{ |\Lambda_{z,n}| \}_{n=1}^{\infty} = \left\{ 1/C_0^{m_z,n} \right\}_{n=1}^{\infty},
\]

such that the segments \([z + \Lambda_{z,n}, z + C_0 \Lambda_{z,n}]\) satisfy the stretching condition (3.3). Here we remind, see (3.4), that

\[
C_0 = 4^{1/a} = 4^{p/(2-s)} \geq 2.
\]

Hence we can associate to every point \( z \in A \) the sequence (4.1) of radii \( \{ |\Lambda_{z,n}| \}_{n=1}^{\infty} \), where \( 1/C_0 < 1 \). Thus we can use Lemma 3.4, with the choice \( \alpha = C_0^2 \), to find a radius

\[
R_{n_0} = 1/C_0^{m_{n_0}}
\]

for which there exist disjoint balls \( \overline{B}(z_j, C_0^2 R_{n_0}) \) such that \( z_j \in A \) and \( R_{n_0} \in \{ |\Lambda_{z,n}| \}_{n=1}^{\infty} \) for every \( j \). Furthermore, again by Lemma 3.4, we can choose \( \epsilon \) and the radius \( R_{n_0} \) as small as we wish.

Then we turn our interest to the disjoint balls \( \overline{B}(z_j, C_0^2 R_{n_0}) \). Fix any of them and denote by \( E_j \) the segment \([z_j + \Lambda_{z_j}, z_j + C_0 \Lambda_{z_j}]\), where \( \Lambda_{z_j} \in \{ |\Lambda_{z,n}| \}_{n=1}^{\infty} \) and \( |\Lambda_{z_j}| = R_{n_0} \), that satisfies the stretching condition (3.3). Such segment exists as \( z_j \in A \) and \( R_{n_0} \in \{ |\Lambda_{z,n}| \}_{n=1}^{\infty} \) for every \( j \). Furthermore, let us define the set \( F_j \) by

\[
F_j = [z_j, z_j + e^{i\pi} C_0^2 \Lambda_{z_j}] \cup \partial B(z_j, C_0^2 R_{n_0}).
\]

For the illustration of these sets, see Figure 2.

Finally, we define \( E = \bigcup_j E_j \) and \( F = \bigcup_j F_j \), and let \( \Gamma \) be the family of paths connecting the sets \( E \) and \( F \). Note that every set \( E_j \) is enclosed by the set \( F_j \), which allows us to think of \( \Gamma \) as the union of the subfamilies \( \Gamma_j \) connecting the sets \( E_j \) and \( F_j \).

Our aim is to use the modulus inequality of Lemma 3.1, and hence we have to estimate the moduli \( M_{K_j}(\Gamma) \) and \( M(f(\Gamma)) \).

Let us start with the modulus \( M_{K_j}(\Gamma) \). To this end, define non-negative Borel-measurable function

\[
\rho_0(z) = \begin{cases} 1/R_{n_0} & \text{if dist}(z, E) < R_{n_0}, \\ 0 & \text{otherwise}. \end{cases}
\]

It is clear that \( \rho_0 \) is admissible with respect to \( \Gamma \) and since \( C_0 \geq 2 \) we see that \( \rho_0 \)
vanishes outside the balls $B_j = B(z_j, C_0^2 R_{no})$. Thus, when $p > 1$, we can estimate

$$M_{K_j} (\Gamma) \leq \int_C K_f(z) \rho_0^2(z) \, dz \leq \left( \int_{B(0,4)} K_f^p(z) \, dz \right)^{1/p} \left( \int_C \rho_0^{2p/(p-1)} \, dz \right)^{(p-1)/p}$$

$$\leq c_{f,p} \left( \left( \frac{1}{R_{no}} \right)^{2p/(p-1)} \pi \left( \frac{C_0^2 R_{no}}{C_{\alpha}^2 \Lambda_{\alpha}} \right)^{\dim(A)-\epsilon} \right)^{(p-1)/p}$$

(4.2)

$$\leq c_{f,p,s} \left( \frac{1}{R_{no}} \right)^{(2+(p-1)(\dim(A)-\epsilon))/p}. $$

On the other hand, if $p = 1$ we can estimate

$$M_{K_j} (\Gamma) \leq \int_C K_f(z) \rho_0^2(z) \, dz \leq \frac{1}{R_{no}^2} \int_{B(0,4)} K_f(z) \, dz \leq c_f \frac{1}{R_{no}^2}. $$

(4.3)

Then we can turn our attention to the modulus $M(f(\Gamma))$. Let us first note that

$$M(f(\Gamma)) = \sum_{j=1}^n M(f(\Gamma_j)), \quad \text{where} \quad n = \left\lfloor \left( \frac{1}{C_0^2 R_{no}} \right)^{\dim(A)-\epsilon} \right\rfloor,$$

is the number of the disjoint balls $B_j$. Here we have used the facts that the sets $f(B_j)$ are disjoint and that every set $f(E_j)$ is enclosed by the set $f(F_j)$, which makes it possible to consider $f(\Gamma)$ as the union of the separate path families $f(\Gamma_j)$.

Let us then concentrate on a single modulus $M(f(\Gamma_j))$. We estimate it from below by first defining a path

$$\tilde{\gamma}_j = f([z_j, z_j + e^{i\pi} C_0^2 \Lambda_{\alpha}]) \cup \gamma_j,$$

where $\gamma_j$ is a path that starts from the point $f(z_j + e^{i\pi} C_0^2 \Lambda_{\alpha})$ and travels to infinity while crossing the set $f(F_j)$ only at the starting point. Then denote by $\tilde{\Gamma}_j$ the path family consisting of the paths connecting the sets $f(E_j)$ and $\tilde{\gamma}_j$. We remind that

$$f(F_j) = f([z_j, z_j + e^{i\pi} C_0^2 \Lambda_{\alpha}]) \cup f(\partial B_j)$$
and emphasize that the path \( f(E_j) \) is enclosed by \( f(F_j) \). Thus, by the monotonicity of the modulus, we see that \( M(f(\Gamma_j)) \geq M(\hat{\Gamma}_j) \).

To estimate the modulus \( M(\hat{\Gamma}_j) \) from below, let us define

\[
x_{j,\text{inf}} = \inf_{z \in E_j} |f(z) - f(z_j)| \quad \text{and} \quad x_{j,\text{sup}} = \sup_{z \in E_j} |f(z) - f(z_j)|.
\]

Since the modulus is conformally invariant we can use translation and stretching to assume that the path \( f(E_j) \) contains the origin and a point with the modulus one, and that the path \( \hat{\gamma}_j \) is unbounded and contains a point with the modulus \( x_{j,\text{inf}}/(x_{j,\text{sup}} - x_{j,\text{inf}}) \). Then it is well known, see [12], Chapter 11, and references therein, that the smallest possible modulus for the path family \( \hat{\Gamma}_j \) occurs when \( f(E_j) = [0,1] \) and \( \hat{\gamma}_j = (-\infty, -x_{j,\text{inf}}/(x_{j,\text{sup}} - x_{j,\text{inf}})] \). In this case the paths \( f(E_j) \) and \( \hat{\gamma}_j \) form the Teichmüller ring, whose modulus can be estimated, using Theorem 7.26 in [13], by

\[
M(\hat{\Gamma}_j) \geq c \log \left( \frac{x_{j,\text{sup}} - x_{j,\text{inf}}}{x_{j,\text{inf}}} \right).
\]

Since the path \( E_j \) satisfies the stretching condition (3.3), with the choice \( a = (2-s)/p \), we can find points \( x, y \in E_j \) for which

\[
|f(y) - f(z_j)| \geq e^{\frac{1}{p} \left( 1/R_{n_0} \right)^{(2-s)/p}} |f(x) - f(z_j)|.
\]

Thus we can continue the above estimate for \( M(\hat{\Gamma}_j) \) and obtain

\[
M(\hat{\Gamma}_j) \geq c \log \left( e^{\frac{1}{p} \left( 1/R_{n_0} \right)^{(2-s)/p}} - 1 \right) \geq \frac{cb}{16} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p},
\]

and hence

\[
M(f(\Gamma_j)) \geq \frac{cb}{16} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p}.
\]

As we can do this estimate for every \( j \) and the stretching condition (3.3) satisfied by the segments \( E_j \) does not depend on the choice of \( j \), we can couple (4.4) with (4.6) to obtain

\[
M(f(\Gamma)) \geq c_{p,b,s} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p + \dim(A) - \epsilon}.
\]

Thus we have found the estimates (4.2), (4.3) and (4.7) for the moduli \( M_{K_j}(\Gamma) \) and \( M(f(\Gamma)) \). When \( p > 1 \) we use the estimates (4.2) and (4.7) together with the modulus inequality (3.1) to obtain

\[
c_{p,b,s} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p + \dim(A) - \epsilon} \leq c_{f,p,s} \left( \frac{1}{R_{n_0}} \right)^{(2+(p-1)(\dim(A)-\epsilon))/p}.
\]

This can be simplified to

\[
\left( \frac{1}{R_{n_0}} \right)^{(\dim(A) - \epsilon)/p} \leq c_{f,p,s,b} \left( \frac{1}{R_{n_0}} \right)^{s/p},
\]
which can hold only if \( \dim(A) \leq s \), as we can choose \( \epsilon \) and the radius \( R_{no} \) as small as we wish.

When \( p = 1 \) we use the estimates (4.3) and (4.7) with the modulus inequality to obtain
\[
c_b,s \left( \frac{1}{R_{no}} \right)^{2-s+\dim(A)-\epsilon} \leq c_f \left( \frac{1}{R_{no}} \right)^2,
\]
which can hold only if \( \dim(A) \leq s \), as we can again choose \( \epsilon \) and the radius \( R_{no} \) as small as we wish. This finishes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

Let us then turn our focus to the rotation and prove Theorem 1.2. Our approach for the proof is similar as in the Theorem 1.1, but the situation is slightly more complicated for the rotation and we will have to work a little bit harder than with the stretching.

Proof of Theorem 1.2. Fix parameters \( p \geq 1, b > 0 \) and \( s \in (0, 2) \), and let \( f : \mathbb{C} \to \mathbb{C} \) be a mapping of finite distortion such that \( K_f(z) \in L^p_{\text{loc}}(\mathbb{C}) \). Moreover, we can again assume without loss of generality that \( A \subset D \). As in the proof of Theorem 1.1, this time using Lemma 3.3 with the choice \( a = \frac{2-s}{p} \), we can find for every point \( z \in A \) a sequence of complex numbers \( \Lambda_{z,n} \), whose moduli form a decreasing sequence
\[
\left\{ |\Lambda_{z,n}| \right\}_{n=1}^{\infty} = \left\{ \frac{1}{C_0 m_{z,n}} \right\}_{n=1}^{\infty},
\]
such that the segments \([z + \Lambda_{z,n}, z + C_0 \Lambda_{z,n}]\) satisfy the rotation condition (3.6).

We continue as in the proof of Theorem 1.1 and use Lemma 3.4, with the choice \( \alpha = C_0^2 \), to find a radius \( R_{no} = 1/C_0^{m_0} \) for which there exist
\[
\left\lfloor \left( \frac{1}{C_0^2 R_{no}} \right)^{\dim(A)-\epsilon} \right\rfloor
\]
disjoint balls \( B(z_j, C_0^2 R_{no}) \) such that \( z_j \in A \) and \( R_{no} \in \{ |\Lambda_{z_j,n}| \}_{n=1}^{\infty} \) for every \( j \). Moreover, again by Lemma 3.4, we recall that \( \epsilon \) and the radius \( R_{no} \) can be chosen as small as we wish.

Then, as before, for any given ball \( B(z_j, C_0^2 R_{no}) \) we denote by \( E_j \) the segment \([z_j + \Lambda_{z_j}, z_j + C_0 \Lambda_{z_j}]\), where \( \Lambda_{z_j} \in \{ \Lambda_{z_j,n} \}_{n=1}^{\infty} \) and \( |\Lambda_{z_j}| = R_{no} \), that satisfies the rotation condition (3.6), and define the set \( F_j \) by
\[
F_j = [z_j, z_j + e^{i\pi} C_0^2 \Lambda_{z_j}] \cup \partial B(z_j, C_0^2 R_{no}).
\]
For the illustration of these sets, see again Figure 2.

Finally, define \( E = \bigcup_j E_j \) and \( F = \bigcup_j F_j \), and let \( \Gamma \) be the family of paths connecting the sets \( E \) and \( F \). Note that the sets \( E \) and \( F \) are defined exactly as
in the proof of Theorem 1.1, and thus we can partition the path family $\Gamma$ to the subfamilies $\Gamma_j$.

Moreover, as the sets $E$ and $F$ are defined as in the proof of Theorem 1.1, we can use the same estimates (4.2) and (4.3) for the modulus $M_{K_j}(\Gamma)$ to obtain

$$M_{K_j}(\Gamma) \leq c_{f,p,s} \left( \frac{1}{R_{n_0}} \right)^{[2+(p-1)(\dim(A) - \epsilon)]/p}, \text{ when } p > 1,$$

and

$$M_{K_j}(\Gamma) \leq c_f \frac{1}{R_{n_0}^2}, \text{ when } p = 1.$$

Hence we can move straight to study the modulus $M(f(\Gamma))$. Here we start with similar arguments as in the pointwise case, see [8] and [9]. Let us first write

$$M(f(\Gamma)) = \inf_{\rho \text{ admissible with respect to } f(\Gamma)} \int \rho^2(z) \, dz = \sum_{j=1}^{n} \inf_{\rho \text{ admissible with respect to } f(\Gamma_j)} \int f(\Gamma_j) \rho^2(z) \, dz,$$

where

$$n = \left\lfloor \left( \frac{1}{C_0 R_{n_0}} \right)^{\dim(A) - \epsilon} \right\rfloor$$

is the number of the disjoint balls $B_j$. Here we have used the facts that the sets $f(B_j)$ are disjoint and that every set $f(E_j)$ is enclosed by the set $f(F_j)$. Let us then concentrate on a single modulus

$$\inf_{\rho \text{ admissible with respect to } f(\Gamma_j)} \int f(\Gamma_j) \rho^2(z) \, dz.$$

We write this integral in the polar form

$$\inf_{\rho \text{ admissible with respect to } f(\Gamma_j)} \int f(\Gamma_j) \rho^2(z) \, dz = \int_0^{2\pi} \int_0^\infty \rho^2(r, \theta) r \, dr \, d\theta,$$

where $r$ is the distance from the point $f(z_j)$ and $\theta$ is the argument with respect to that point. Clearly we can assume without the loss of regularity that $\rho(r, \theta) = 0$ if the point represented by the coordinates $r, \theta$ does not lie in the set $f(B_j)$. Our aim is to provide a lower bound for the integral

$$\int_0^\infty \rho^2(r, \theta) r \, dr$$

that holds for an arbitrary direction $\theta \in [0, 2\pi)$ and an arbitrary admissible $\rho$.

We estimate the integral (5.6) from below with a similar technique as in the pointwise case by first finding many disjoint line segments, for which one endpoint lies in $f(E_j)$ and the other in $f(F_j)$, and then using admissibility of $\rho$ to estimate the integral of $\rho^2(r, \theta) r$ over these line segments. The key idea is to note that the
sets \( f(E_j) \) and \( f(F_j) \) must cycle around the point \( f(z_j) \) alternately, see Figure 3 for illustration.

To this end, fix a direction \( \theta \) and let \( L_\theta \) be the half-line starting from the point \( f(z_j) \) to the direction \( \theta \). Then choose points \( t_1, t_3 \in E_j \), for which \(|t_1 - z_j| < |t_3 - z_j|\) and \( f(t_3) \in L_\theta \), such that the path \( f(E_j) \) winds once around the point \( f(z_j) \) when \( z \) travels from \( t_3 \) to \( t_1 \) along the segment \( E_j \). Since the mapping \( f \) is a homeomorphism and the set \( f(F_j) \) contains the point \( f(z_j) \) and the boundary \( f(\partial B_j) \), the set \( f(F_j) \) must intersect the line \( (f(t_1), f(t_3)) \) at least once, let us say at the point \( f(t_2) \), where \( t_2 \in F_j \). Moreover, we can choose the points \( t_1 \) and \( t_2 \) such that there are no points from the sets \( f(E_j) \) or \( f(F_j) \) in the line segment \( (f(t_1), f(t_2)) \). These line segments \( [f(t_1), f(t_2)] \subset L_\theta \) are the ones we are looking for.

As the path \( f(E_j) \) has a subpath which cycles around the point \( f(z_j) \) at least
\[
\left\lfloor \frac{b}{16 \cdot 2\pi} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p} \right\rfloor \quad \text{times},
\]
due to the rotation condition (3.6) of the line segment \( E_j \), the number of the disjoint line segments obtained this way is at least
\[
m(R_{n_0}) = \left\lfloor \frac{b}{16 \cdot 2\pi} \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p} \right\rfloor - 1 \geq c_b \left( \frac{1}{R_{n_0}} \right)^{(2-s)/p}.
\]

We emphasize that (5.7) does not depend on the direction \( \theta \), but gives a lower bound for the amount of desired line segments for an arbitrary direction \( \theta \).

Write these line segments in the form \([f(z_j) + a_l e^{il\theta}, f(z_j) + b_l e^{il\theta}]\), where \( l \in \{1, 2, \ldots, m(R_{n_0})\} \). Since these line segments are disjoint, the distances \( a_l \) and \( b_l \) satisfy
\[
x_{j,\text{inf}} \leq a_1 < b_1 < \cdots < a_{m(R_{n_0})} < b_{m(R_{n_0})} < x_{j,\text{sup}},
\]
where we recall that
\[
x_{j,\text{inf}} = \inf_{z \in E_j} |f(z) - f(z_j)| \quad \text{and} \quad x_{j,\text{sup}} = \sup_{z \in E_j} |f(z) - f(z_j)|.
\]
Hence we can estimate (5.6) by

\[(5.8) \quad \int_0^\infty \rho^2(r, \theta) \, r \, dr \geq \sum_{i=1}^{m(R_{n_0})} \int_{a_i}^{b_i} \rho^2(r, \theta) \, r \, dr.\]

Then note that every line segment \([f(z_j) + a_ie^{i\theta}, f(z_j) + b_ie^{i\theta}]\) belongs to the path family \(f(\Gamma_j)\), since the endpoints of such line segment are in the different sets \(f(E_j)\) and \(f(F_j)\). Thus we can estimate the integral over any such line segment, using the reverse Hölder inequality and the admissibility of \(\rho\), by

\[\int_{a_i}^{b_i} \rho^2(r, \theta) \, r \, dr \geq \left( \int_{a_i}^{b_i} \rho(r, \theta) \, dr \right)^2 \left( \int_{a_i}^{b_i} \frac{1}{r} \, dr \right)^{-1} \geq \frac{1}{\log(b_i/a_i)}.\]

We combine this with (5.8) to obtain

\[(5.9) \quad \int_0^\infty \rho^2(r, \theta) \, r \, dr \geq \sum_{l=1}^{m(R_{n_0})} \frac{1}{\log(b_l/a_l)}.\]

In order to estimate this further, we use the arithmetic-harmonic means inequality

\[(5.10) \quad \sum_{l=1}^{m} x_l \geq \frac{m^2}{\sum_{l=1}^{m} \frac{1}{x_l}},\]

which holds whenever every \(x_l\) is a positive real number.

First we must modify (5.9) by

\[(5.11) \quad \sum_{l=1}^{m(R_{n_0})} \frac{1}{\log(b_l/a_l)} \geq \log(a_2/x_{j,\text{inf}}) + \sum_{l=2}^{m(R_{n_0})-1} \frac{1}{\log(a_l+1/a_l)} + \frac{1}{\log(x_{j,\text{sup}}/a_{m(R_{n_0})})}.\]

Then we use the inequality (5.10), with the choices

\[x_1 = \frac{1}{\log(a_2/x_{j,\text{inf}})}, \quad x_l = \frac{1}{\log(a_l+1/a_l)}, \quad \text{when} \ l \in \{2, 3, \ldots, m(R_{n_0}) - 1\},\]

and

\[x_{m(R_{n_0})} = \frac{1}{\log(x_{j,\text{sup}}/a_{m(R_{n_0})})}\]

to continue the estimate (5.11) by

\[\sum_{l=1}^{m(R_{n_0})} \frac{1}{\log(b_l/a_l)} \geq \frac{m^2(R_{n_0})}{\log(a_2/x_{j,\text{inf}}) + \sum_{l=2}^{m(R_{n_0})-1} \log(a_l+1/a_l) + \log(x_{j,\text{sup}}/a_{m(R_{n_0})})} = \frac{m^2(R_{n_0})}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})}.\]
This estimate together with (5.9) gives

\[
\int_0^\infty \rho^2(r, \theta) r \, dr \geq \frac{m^2(R_{r_0})}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})}.
\]

(5.12)

Note that the estimate (5.12) does not depend on the direction \( \theta \) or on the function \( \rho \), which is admissible with respect to the path family \( f(\Gamma_j) \). Thus we can estimate

\[
\inf_{\rho \text{ admissible}} \int_0^{2\pi} \int_0^\infty \rho^2(r, \theta) r \, dr \, d\theta \geq \frac{c \cdot m^2(R_{r_0})}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})}.
\]

(5.13)

Since we can do this estimate for every \( j \) and the rotation condition satisfied by the segments \( E_j \) does not depend on the choice of \( j \), we can combine (5.13) with (5.4) to obtain

\[
M(f(\Gamma)) \geq c_b \left( \frac{1}{R_{r_0}} \right)^{2(2-s)/p} \sum_{j=1}^{n} \frac{1}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})}.
\]

(5.14)

Thus we have found the estimates (5.2), (5.3) and (5.14) for the moduli \( M_{K_j}(\Gamma) \) and \( M(f(\Gamma)) \). When \( p > 1 \), we use the estimates (5.2) and (5.14) with the modulus inequality (3.1) to obtain

\[
\left( \frac{1}{R_{r_0}} \right)^{2(2-s)/p} \sum_{j=1}^{n} \frac{1}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})} \leq c_{f,p,s,b} \left( \frac{1}{R_{r_0}} \right)^{(2+(p-1)(\dim(A)-\epsilon))/p}.
\]

(5.15)

If \( p = 1 \), we use the estimates (5.3) and (5.14) with the modulus inequality to obtain

\[
\left( \frac{1}{R_{r_0}} \right)^{2(2-s)} \sum_{j=1}^{n} \frac{1}{\log(x_{j,\text{sup}}/x_{j,\text{inf}})} \leq c_{f,b} \left( \frac{1}{R_{r_0}} \right)^2.
\]

(5.16)

Unfortunately, here our proof must differ from the method used in [9], as the pointwise modulus of continuity result, see (1.1), would be too crude tool for estimating the stretch of the mapping \( f \) at the points \( z_j \). Neither can we use Theorem 1.1 as we have made no assumptions on the stretching of the paths \( f(E_j) \). Thus we have to work some more and use the ideas from the proof of Theorem 1.1 to estimate the stretching terms \( \log(x_{j,\text{sup}}/x_{j,\text{inf}}) \), which together with the above inequalities will prove Theorem 1.2.

Here we will encounter a technical obstacle in the form of ensuring that

\[
x_{j,\text{sup}} - x_{j,\text{inf}}
\]

is not too small, which we require to utilize a similar estimate as in (4.5). For overcoming this, we show that if there exists a fixed constant \( 0 < c < 1 \) such that we can find

\[
c \left\lfloor \left( \frac{1}{C_0 R_{r_0}} \right)^{\dim(A)-\epsilon} \right\rfloor
\]

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\[
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\]

is not too small, which we require to utilize a similar estimate as in (4.5). For overcoming this, we show that if there exists a fixed constant \( 0 < c < 1 \) such that we can find

\[
c \left\lfloor \left( \frac{1}{C_0 R_{r_0}} \right)^{\dim(A)-\epsilon} \right\rfloor
\]
balls $B_j$ for which
\[ x_{j,\text{sup}} < e \cdot x_{j,\text{inf}}, \]
then the inequalities (5.15) and (5.16) in fact suffice for proving Theorem 1.2. This is not surprising, since in order to have strong rotation one expects to also have strong stretching, just as in the pointwise case, see [8] and [9].

Let us first consider the case $p > 1$. The above assumption for $x_{j,\text{inf}}$ and $x_{j,\text{sup}}$ coupled with the inequality (5.15) gives
\[
\left( \frac{1}{R_{n_0}} \right)^{2(2-s)/p} \left( \frac{1}{C_0^2 R_{n_0}} \right)^{\dim(A) - \epsilon} \leq c f, p, s, b \left( \frac{1}{R_{n_0}} \right)^{(2+(p-1)(\dim(A) - \epsilon))/p}.
\]
This can be simplified to
\[
\left( \frac{1}{R_{n_0}} \right)^{2-2s+\dim(A) - \epsilon} \leq c f, p, s, b,
\]
which can clearly hold only if $\dim(A) \leq 2s - 2 \leq s$, since we can choose $\epsilon$ and the radius $R_{n_0}$ as small as we wish and $s \in (0, 2)$. The case $p = 1$ is shown in a completely similar manner.

Thus we can assume that, for a fixed constant $0 < c < 1$, there exist
\[
e \left[ \left( \frac{1}{C_0^2 R_{n_0}} \right)^{\dim(A) - \epsilon} \right] \]
balls $B_j$ such that $x_{j,\text{sup}} \geq e \cdot x_{j,\text{inf}}$. It is clear that the constant $c$ plays no role in the proof, especially since we can choose $\epsilon$ as small as we wish.

This in mind, we can continue our proof and proceed to estimate the stretching terms $\log \left( x_{j,\text{sup}} / x_{j,\text{inf}} \right)$. To this end we use the modulus inequality (3.1) with the same path family $\Gamma$ and the same estimate for the modulus $M_{K_f}(\Gamma)$, while estimating the modulus $M(f(\Gamma))$ in a similar manner as in the proof of Theorem 1.1.

So, we once more write
\[
M(f(\Gamma)) = \sum_{j=1}^n M(f(\Gamma_j)),
\]
and continue as in the proof of Theorem 1.1 to obtain
\[
M(f(\Gamma_j)) \geq c \log \left( \frac{x_{j,\text{sup}} - x_{j,\text{inf}}}{x_{j,\text{inf}}} \right).
\]
Then we use the assumption $x_{j,\text{sup}} \geq e \cdot x_{j,\text{inf}}$ and continue the above estimate by
\[
M(f(\Gamma)) \geq c \log \left( \frac{e - 1}{e} \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right) \geq c \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right).
\]
And as we go through every ball $B_j$ and use the estimate (5.17), we obtain
\[
M(f(\Gamma)) \geq c \sum_{j=1}^n \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right).
\]
Then we use the modulus inequality (3.1) with the modulus estimates (5.2), (5.3) and (5.18), which gives, when \( p > 1 \),

\[
\sum_{j=1}^{n} \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right) \leq c_{f,p,s} \left( \frac{1}{R_{n_0}} \right)^{2+[(p-1)(\dim(A)-\epsilon)]/p},
\]

and, when \( p = 1 \),

\[
\sum_{j=1}^{n} \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right) \leq c_{f} \left( \frac{1}{R_{n_0}} \right)^{2}.
\]

Finally, we use the inequalities (5.15), (5.16), (5.19) and (5.20) to finish the proof. To this end, note that any positive real numbers \( x_1, \ldots, x_m \) satisfy

\[
(x_1 + \cdots + x_m) \left( \frac{1}{x_1} + \cdots + \frac{1}{x_m} \right) \geq m^2.
\]

Then choose \( x_j = \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right) \) for every \( j \in \{1, 2, \ldots, n\} \), where we remind that

\[
n = c \left( \frac{1}{C_d R_{n_0}} \right)^{\dim(A)-\epsilon},
\]

to obtain

\[
(5.21) \quad c \left( \frac{1}{C_d R_{n_0}} \right)^{2(\dim(A)-\epsilon)} \leq \sum_{j=1}^{n} \log \left( \frac{x_{j,\text{sup}}}{x_{j,\text{inf}}} \right) \sum_{j=1}^{n} \frac{1}{x_{j,\text{sup}}/x_{j,\text{inf}}}.
\]

Now assume that \( p > 1 \) and use the inequalities (5.15) and (5.19) to estimate the sums in (5.21), and simplify to obtain

\[
\left( \frac{1}{R_{n_0}} \right)^{2(\dim(A)-\epsilon)} \leq c_{f,p,s,b} \left( \frac{1}{R_{n_0}} \right)^{4+2[(p-1)(\dim(A)-\epsilon)-2(2-s)]/p} \iff \left( \frac{1}{R_{n_0}} \right)^{2(\dim(A)-\epsilon)/p} \leq c_{f,p,s,b} \left( \frac{1}{R_{n_0}} \right)^{2s/p}.
\]

But this can hold only if \( \dim(A) \leq s \), since we can choose \( \epsilon \) and the radius \( R_{n_0} \) as small as we wish. Hence the proof is finished in the case \( p > 1 \).

When \( p = 1 \) we continue from (5.21) and use the inequalities (5.16) and (5.20) to obtain

\[
\left( \frac{1}{R_{n_0}} \right)^{2(\dim(A)-\epsilon)} \leq c_{f,s,b} \left( \frac{1}{R_{n_0}} \right)^{4-2(2-s)} = c_{f,s,b} \left( \frac{1}{R_{n_0}} \right)^{2s},
\]

which can hold only if \( \dim(A) \leq s \), since \( \epsilon \) and the radius \( R_{n_0} \) can be chosen arbitrary small. This completes the proof of Theorem 1.2. \( \square \)
6. Sharpness of Theorems 1.1 and 1.2, and an application

Let us then show that Theorems 1.1 and 1.2 are optimal in the sense of Theorem 1.4.

To this end, we first show that Theorem 1.4 follows from the following result.

Theorem 6.1. Let $p \geq 1$, $s \in (0, 2)$ and $\epsilon > 0$ be given. Then we can find a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $p$-integrable distortion and a set $A \subset \mathbb{D}$, for which $\dim(A) = s$, such that for every point $z \in A$ there exist a branch of the argument and a sequence $\lambda_{z,n}$, where $|\lambda_{z,n}| \to 0$, satisfying

\begin{equation}
|f(z + \lambda_{z,n}) - f(z)| \leq e^{-(1/|\lambda_{z,n}|)(2-s-\epsilon)/p}
\end{equation}

and

\begin{equation}
|\arg(f(z + \lambda_{z,n}) - f(z))| \geq \left( \frac{1}{|\lambda_{z,n}|} \right)^{(2-s-\epsilon)/p},
\end{equation}

for every $n$. Moreover, we can choose the mapping $f$ such that $f(z) = z$ when $z \notin \mathbb{D}$.

To see this, note that for any given $p \geq 1$ and $s_1 \in (0, 2)$ we can fix $s = s_1 - \epsilon$, where $\epsilon > 0$ is arbitrary small. Then we use Theorem 6.1 to find a mapping $f_\epsilon$ and a set $A_\epsilon \subset \mathbb{D}$, with $\dim(A_\epsilon) = s$, such that the conditions (6.1) and (6.2) hold for every point $z \in A_\epsilon$ with the exponent $(2-s_1)/p$. As the mapping $f_\epsilon$ coincides with the identity mapping outside of the unit disc, we can choose a sequence $\epsilon_n \to 0$ and glue the corresponding mappings $f_\epsilon_n$ together to construct the limit map $f : \mathbb{C} \to \mathbb{C}$, which is a mapping of finite distortion with $K_f(z) \in L^p_{\text{loc}}(\mathbb{C})$. Moreover, the mapping $f$ satisfies the conditions (6.1) and (6.2), with the exponent $(2-s_1)/p$, for every point in the set

$A = \bigcup_{n=1}^{\infty} A_{\epsilon_n},$

where $\dim(A) = s_1$. Thus Theorem 1.4 follows from Theorem 6.1.

To construct the mappings $f$ in Theorem 6.1, we use families of nested annuli, formed in an iterative manner, whose inner discs create a Cantor set at the limit. This approach is quite classical for establishing extremal examples for quasiconformal mappings, see, for example, [2] and [11]. For mappings with $p$-integrable distortion similar construction was used, for example, in [3].

On the first level we choose $M$ balls $B_{1,i}$, where $M$ is a big integer we fix later and $i \in \{1, 2, \ldots, M\}$, with the radius

$r = \frac{1}{M^{1/s}}.$

We pack them inside the unit disk in such a manner that the distance between the center points is at least $2er$, and that the distance between the boundary $\partial D$ and the center points is also at least $2er$. Moreover, we arrange them so that the
center points create a square grid, see Figure 4. Note that if we choose $M = m^2$ big enough this can be done for an arbitrary $s \in (0, 2)$.

Then we create annuli out of the balls $B_{1,i}$ by defining $A_{1,i} = eB_{1,i} \setminus B_{1,i}$, and check that they are disjoint and stay inside the unit disc. Denote the family of the first level balls $B_{1,i}$ by $\mathbb{B}_1$ and the family of the outer balls of the first level annuli $A_{1,i}$ by $\mathbb{A}_1$.

Let $\phi_i : \mathbb{D} \to B_{1,i}$ be a similarity. We define the family of the second level balls $\mathbb{B}_2$ by

$$\mathbb{B}_2 = \bigcup_{i=1}^{M} \phi_i(\mathbb{B}_1),$$

and the family of the second level outer balls by

$$\mathbb{A}_2 = \bigcup_{i=1}^{M} \phi_i(\mathbb{A}_1).$$

These families create the second level annuli, which lie inside the first level balls $\mathbb{B}_1$, with the inner radius $r^2$ and the outer radius $er^2$.

We continue in this manner and define

$$A = \bigcap_{n=1}^{\infty} \bigcup_{B \in \mathbb{B}_n} B,$$

which is a self similar Cantor set of the plane. The Hausdorff dimension of the set $A$ can be computed to be $s$ by checking that it satisfies the equation

$$Mr^s = 1.$$
For an arbitrary annulus $B \setminus \frac{1}{e} B$, with $B = B(a, R)$, and an arbitrary $K \geq 1$, we define the radial map

$$\psi_{B,K}(z) = \begin{cases} z & \text{if } z \notin B(a, R), \\ a + R \frac{z - a}{|z - a|} |(1 + i)K| & \text{if } z \in B(a, R) \setminus B(a, R/e), \\ a + e^{-K+1} e^{-iK} (z - a) & \text{if } z \in B(a, R/e), \end{cases}$$

which will serve as the building block for the mapping $f$. Note that the map $\psi_{B,K}(z)$ is $cK$ quasiconformal, where $c$ is some fixed constant, and conformal outside the annulus $B \setminus \frac{1}{e} B$. Let us then define

$$K_n = M_n^{p(2-s-\epsilon)/s},$$

which we will use as the distortion at the corresponding step of the construction, and calculate

$$\sum_{n=1}^{\infty} |A_n| K_n^p \leq c \sum_{n=1}^{\infty} M_n^{2n} K_n^p = c \sum_{n=1}^{\infty} \frac{M_n}{M_n^{1/s}} M_n^{n(2-s-\epsilon)/s}$$

$$= c \sum_{n=1}^{\infty} \frac{1}{M_n^{1/s}} < \infty.$$

With these preparations, let us begin the construction of the mapping $f$. As the first step we define

$$f_1(z) = \begin{cases} z & \text{if } z \notin \bigcup_{B \in A_1} B, \\ \psi_{B_1,K_1}(z) & \text{if } z \in B \text{ with } B \in A_1. \end{cases}$$

Then we assume that the previous step $f_{n-1}$ has been defined and set

$$f_n(z) = \begin{cases} f_{n-1}(z) & \text{if } z \notin \bigcup_{B \in A_n} B, \\ \psi_{f_{n-1}(B),K_n}(f_{n-1}(z)) & \text{if } z \in B \text{ with } B \in A_n. \end{cases}$$

Since $f_{n-1}$ is conformal inside the family $A_n$, we see that $f_n$ is $cK_i$-quasiconformal inside the family of the level $i$ annuli, where $i \in \{1, 2, \ldots, n\}$, and conformal outside these annuli.

From the definition of the mappings $f_n$ it is clear that $f_{n+1}(z) \neq f_n(z)$ only inside the family $A_{n+1}$, and hence it is easy to see that the mappings $f_n$ form a Cauchy sequence. Thus there exists the limit map

$$f = \lim_{n \to \infty} f_n,$$

which is clearly a homeomorphism. Moreover, the mapping $f$ is differentiable almost everywhere as it is clearly differentiable outside the set $A$ and the boundaries of the annuli $A_{n,i}$. From the definitions of the mapping $f$ and the radial mapping $\psi_{B,K}$, see (6.4), we can calculate that

$$|f_z(z)| \leq c K_n$$
if $z$ lies inside an annulus from the level $n$. Moreover, we see that $|Df(z)| \leq 1$ if $f$ is differentiable at a point $z$ which does not lie inside any annulus $A_{n,i}$. Furthermore, since $|Df(z)| \leq c |f_z(z)|$, the estimate (6.7) together with the calculation (6.6) imply that $Df(z) \in L^1_{\text{loc}}(\mathbb{C})$.

Next we note that as we chose the balls $B_{1,i}$ arranged in a square grid, see Figure 4, it follows that for almost every line parallel to the coordinate axes we can find a neighbourhood where the mapping $f$ coincides with the quasiconformal mappings $f_n$, for every big enough $n$. Thus the mapping $f$ is absolutely continuous for almost every line parallel to the coordinate axes. This together with $Df(z) \in L^1_{\text{loc}}(\mathbb{C})$ implies that $f \in W^{1,1}_{\text{loc}}(\mathbb{C})$. Furthermore, since $f$ is a homeomorphism, this also shows that $J_f(z) \in L^1_{\text{loc}}(\mathbb{C})$. Thus the mapping $f$ is a mapping of finite distortion, with the distortion

$$K_f(z) = \begin{cases} cK_n & \text{if } z \in A_{n,i} \text{ for some } i \in \{1, 2, \ldots, M^n\}, \\ 1 & \text{otherwise,} \end{cases}$$

that is $p$-integrable due to the estimate (6.6).

Hence, the only thing left to show is that the mapping $f$ satisfies the stretching condition (6.1) and the rotation condition (6.2) at every point $z \in A$.

Let us first verify the rotation condition. To this end, choose an arbitrary point $z \in A$. For any such point there exists the unique sequence of nested balls $B_{n,i,n}$ such that

$$z = \bigcap_{n=1}^{\infty} B_{n,i_n}.$$

Since the point $z$ lies inside the balls $B_{n,i_n}$ we can find at every step $n$ a complex number $\lambda_{z,n}$ that satisfies $z + \lambda_{z,n} \in \partial B_{n,i_n}$ and

$$|\lambda_{z,n}| = r(B_{n,i_n}).$$

Next, fix an arbitrary branch of the argument and denote

$$\alpha = \max_{\theta \in [0,2\pi]} |\arg(f(z + e^{i\theta}) - f(z))|.$$

Then, using the definition of the mapping $f$ and the definition (6.4) of the radial map $\psi_{B,K}$, we will estimate the argument

$$|\arg(f(z + \lambda_{z,n}) - f(z))|$$

as in [7]. Namely, we first sum up the rotation coming from crossing the annuli $A_{j,i,j}$, which can be calculated from (6.4) to be of order $K_n$ for every $j \in \{1, 2, \ldots, n\}$, and adjust this with an error term at every step. This error term appears as the point $z$ does not have to be the center point of these annuli and as we might not travel through these annuli in a radial line. However, it is easy to see that for any given annuli the modulus of this error term is at most $4\pi$.

The rest of the rotation, which we also view as an error term, comes from crossing the line segment connecting the point $f(z + \lambda_{z,n}/|\lambda_{z,n}|)$ to the annulus $A_{1,i_1}$ and
from crossing the line segments between the annuli $A_{j,i}$, where $j \in \{1, 2, \ldots, n\}$. As the image of any of these line segments under the mapping $f$ might as well be a line segment from the arguments perspective, we see that the combined rotation coming from these crossings has modulus smaller than $n\pi$.

All in all, we get that the error term when estimating the argument (6.8) has the modulus smaller than $5\pi n$, and hence we obtain

$$|\arg(f(z + \lambda_{z,n}) - f(z))| \geq \arg(e^{iK_n}) + \sum_{j=1}^{n-1} \arg(e^{iK_j}) - \alpha - 5\pi n \geq M \frac{5(2-s-\epsilon)}{p},$$

whenever $n$ is big enough. This is enough to prove the claim concerning the rotation.

Let us then check that the sequence $\lambda_{z,n}$ satisfies also the stretching condition. First we note that since $z + \lambda_{z,n} \in \partial B_{n,1}$, we get

$$f(z + \lambda_{z,n}) = f_n(z_j + \lambda_{z,n}).$$

Moreover, we can choose the parameter $M$ as big as we want, which ensures that we can assume that

$$r(B_{n,1}) \geq 100 \cdot r(B_{n+1,1}+1).$$

Thus, since $f(z) \in f_n(B_{n+1,1}+1)$, the modulus $|\lambda_{z,n}| = r(B_{n,1})$ is big compared to the radius $r(B_{n+1,1}+1)$ and $f_n$ maps the ball $B_{n,1}$ as a similarity mapping, we can estimate that

$$|f(z + \lambda_{z,n}) - f(z)| < 2|f_n(z + \lambda_{z,n}) - f_n(z)|.$$

And hence we can estimate

$$|f(z + \lambda_{z,n}) - f(z)| < 2e^{-K_n+1}|\lambda_{z,n}| \leq e^{-M \frac{5(2-s-\epsilon)}{p}} = e^{-(1/|\lambda_{z,n}|)(2-s-\epsilon)/p},$$

for small enough $|\lambda_{z,n}|$. This verifies the stretching condition and finishes the proof of Theorem 6.1, and thus also proves Theorem 1.4. Hence Theorems 1.1 and 1.2 are sharp even in the sense of the joint rotational and stretching multifractal spectra.

### 6.1. Application to area contraction

To finish, we use Theorem 1.1 to prove the optimal bound for the area contraction of mappings with $p$-integrable distortion. The idea of the proof is similar to that in [3], but we take advantage of the better stretching estimates given by Theorem 1.1.

To this end, let $f : \mathbb{C} \to \mathbb{C}$ be a mapping with $p$-integrable distortion and fix $s \in (0, 2)$ and $\epsilon > 0$. We aim to show that every set $A \subset \mathbb{C}$ for which $H^h(f(A)) = 0$, where the gauge function is defined by

$$h(t) = \left(\frac{1}{\log(1/t)}\right)^{ps/(2-(s-\epsilon))},$$

satisfies $H^s(A) = 0$. 

Proof of Theorem 1.5. Fix an arbitrary set \( A \subset \mathbb{C} \) such that \( H^b(f(A)) = 0 \). Let \( A_1 \subset A \) consist of those points \( z \in A \) for which there exists a sequence of complex numbers \( \lambda_{z,n} \), such that the moduli \( |\lambda_{z,n}| \to 0 \) form a decreasing sequence, satisfying

\[
|f(z + \lambda_{z,n}) - f(z)| \leq e^{-(1/|\lambda_{z,n}|^{2-(s-\epsilon)})/p}.
\]

Due to Theorem 1.1, we know that \( \dim(A_1) \leq s - \epsilon \), and hence \( H^s(A_1) = 0 \). Thus we can concentrate on estimating the size of the set \( A_2 = A \setminus A_1 \).

Since we cannot find sequences \( \lambda_{z,n} \) satisfying (6.10) for the points \( z \in A_2 \), we know that for every such point there exists a radius \( r_z \) such that

\[
|f(z + h) - f(z)| \geq e^{-(1/|h|^{2-(s-\epsilon)})/p},
\]

when \( |h| < r_z \). Thus for every point \( x = f(z) \) there exists a radius \( r_x \) such that

\[
|f^{-1}(x + h) - f^{-1}(x)| \leq \left( \frac{1}{\log(1/|h|)} \right)^{ps/(2-(s-\epsilon))}
\]

when \( |h| < r_x \).

As we assumed that \( H^b(f(A)) = 0 \), it follows that \( H^b(f(A_2)) = 0 \). Hence we can cover the set \( f(A_2) \) with balls \( B(x_i, r_i) \) such that \( x_i \in f(A_2) \), the radii \( r_i < r_{x_i} \) can be chosen arbitrary small and

\[
\sum_i h(2r_i) < \epsilon_1,
\]

for any given \( \epsilon_1 > 0 \). Then we can estimate the size of the set \( A_2 \) by covering it with the sets \( f^{-1}(B(x_i, r_i)) \), whose diameter can be controlled by (6.11). So we calculate

\[
\sum_i (\text{diam}(f^{-1}(B(x_i, r_i))))^s \leq 2^s \sum_i \left( \frac{1}{\log(1/r_i)} \right)^{ps/(2-(s-\epsilon))} \leq 2^s \sum_i \left( \frac{1}{\log(1/(2r_i))} \right)^{ps/(2-(s-\epsilon))} = 2^s \sum_i h(2r_i) < 4\epsilon_1,
\]

where in the last inequality we have used the assumption (6.12). Thus we see that \( H^s(A_2) = 0 \) by letting \( \epsilon_1 \to 0 \). And since \( A = A_1 \cup A_2 \), where both \( H^s(A_1) = 0 \) and \( H^s(A_2) = 0 \), we conclude that \( H^s(A) = 0 \), which finishes the proof.

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