NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

European Mathematical Society

25 years

March 2015
Issue 95
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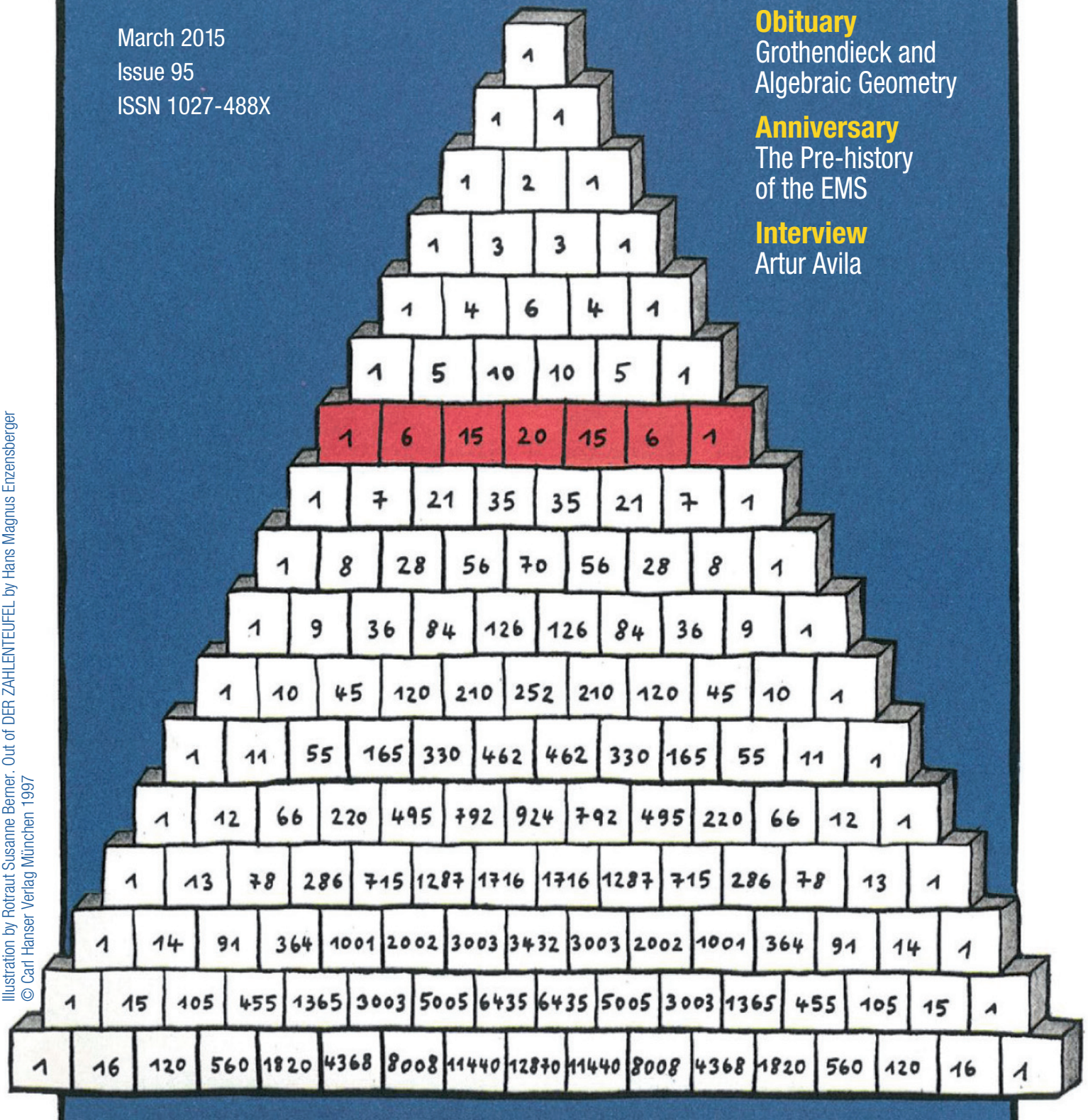
Features
The *h*-Principle and Onsager's Conjecture
Cannons at Sparrows

Obituary
Grothendieck and Algebraic Geometry

Anniversary
The Pre-history of the EMS

Interview
Artur Avila



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Journals published by the

European Mathematical Society

25 years

ISSN print 2308-5827
ISSN online 2308-5835
2015. Vol. 2. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
238 € print+online

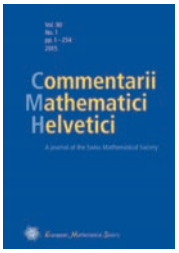
Annales de l'Institut Henri Poincaré D

Editors-in-Chief:

Gérard H. E. Duchamp (Université Paris XIII, France)
Vincent Rivasseau (Université Paris XI, France)
Alan Sokal (New York University, USA and University College London, UK)

Aims and Scope

The journal is dedicated to publishing high-quality original research articles and survey articles in which combinatorics and physics interact in both directions. Combinatorial papers should be motivated by potential applications to physical phenomena or models, while physics papers should contain some interesting combinatorial development.



ISSN print 0010-2571
ISSN online 1420-8946
2015. Vol. 90. 4 issues
Approx. 1000 pages. 17.0 x 24.0 cm
Price of subscription:
328 € online only
388 € print+online

A journal of the Swiss Mathematical Society

Editor-in-Chief:

Eva Bayer-Fluckiger (École Polytechnique Fédérale de Lausanne, Switzerland)

Aims and Scope

The *Commentarii Mathematici Helvetici* was established on the occasion of a meeting of the Swiss Mathematical Society in May 1928, the first volume published in 1929. The journal soon gained international reputation and is one of the world's leading mathematical periodicals. The journal is intended for the publication of original research articles on all aspects in mathematics.

ISSN print 0013-8584
ISSN online 2309-4672
2015. Vol. 61. 2 issues
Approx. 450 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
238 € print+online

Official organ of The International Commission on Mathematical Instruction

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Aims and Scope

L'Enseignement Mathématique was founded in 1899 by Henri Fehr (Geneva) and Charles-Ange Laisant (Paris). It is intended primarily for publication of high-quality research and expository papers in mathematics. Approximately 60 pages each year will be devoted to book reviews.

ISSN print 0032-5155
ISSN online 1662-2758
2015. Vol. 72. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
196 € online only
236 € print+online

A journal of the Portuguese Mathematical Society

Editor-in-Chief:

Luis Nunes Vicente (Universidade de Coimbra, Portugal)

Aims and Scope

Since its foundation in 1937, *Portugaliae Mathematica* has aimed at publishing high-level research articles in all branches of mathematics. With great efforts by its founders, the journal was able to publish articles by some of the best mathematicians of the time. In 2001 a *New Series of Portugaliae Mathematica* was started, reaffirming the purpose of maintaining a high-level research journal in mathematics with a wide range scope.

ISSN print 0034-5318
ISSN online 1663-4926
2015. Vol. 51. 4 issues
Approx. 800 pages. 17.0 x 24.0 cm
Price of subscription:
298 € online only
348 € print+online

A journal of the Research Institute for Mathematical Sciences of Kyoto University

Editor-in-Chief:

S. Mochizuki

Aims and Scope

The aim of the *Publications of the Research Institute for Mathematical Sciences* is to publish original research papers in the mathematical sciences. Occasionally surveys are included at the request of the editorial board.

ISSN print 0213-2230
ISSN online 2235-0616
2015. Vol. 31. 4 issues
Approx. 1500 pages. 17.0 x 24.0 cm
Price of subscription:
368 € online only
428 € print+online

A scientific journal of the Real Sociedad Matemática Española

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Aims and Scope

Revista Matemática Iberoamericana publishes original research articles on all areas of mathematics. Its distinguished Editorial Board selects papers according to the highest standards. Founded in 1985, *Revista* is a scientific journal of Real Sociedad Matemática Española.

European Mathematical Society Publishing House
Seminar for Applied Mathematics, ETH-Zentrum SEW A27
Scheuchzerstrasse 70
CH-8092 Zürich, Switzerland
subscriptions@ems-ph.org
www.ems-ph.org

Editorial Team

Editor-in-Chief

Lucia Di Vizio
Université de Versailles-St Quentin
Laboratoire de Mathématiques
45 avenue des États-Unis
78035 Versailles cedex, France
e-mail: divizio@math.cnrs.fr

Copy Editor

Chris Nunn
119 St Michaels Road,
Aldershot, GU12 4JW, UK
e-mail: nunn2quick@gmail.com

Editors

Jean-Paul Allouche
(Book Reviews)
CNRS, IMJ-PRG, Équipe Combinatoire et Optimisation
Université Pierre et Marie Curie
4, Place Jussieu, Case 247
75252 Paris Cedex 05, France
e-mail: jean-paul.allouche@imj-prg.fr

Jorge Buescu
(Societies)
Dep. Matemática, Faculdade de Ciências, Edifício C6,
Piso 2 Campo Grande
1749-006 Lisboa, Portugal
e-mail: jbuescu@ptmat.fc.ul.pt

Jean-Luc Dorier
(Math. Education)
FPSE – Université de Genève
Bd du pont d'Arve, 40
1211 Genève 4, Switzerland
bartolini@unimore.it

Eva-Maria Feichtner
(Research Centres)
Department of Mathematics
University of Bremen
28359 Bremen, Germany
e-mail: emf@math.uni-bremen.de

Javier Fresán
(Young Mathematicians' Column)
Departement Mathematik
ETH Zürich
Rämistrasse 101, HG J 65
8092 Zürich, Switzerland
e-mail: javier.fresan@math.ethz.ch

Scan the QR code to go to the Newsletter web page:
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Vladimir R. Kostic
(Social Media)
Department of Mathematics and Informatics
University of Novi Sad
21000 Novi Sad, Serbia
e-mail: vladimir.slk@gmail.com

Eva Miranda
Departament de Matemàtica Aplicada I, EPSEB, Edifici P
Universitat Politècnica de Catalunya
Av. del Dr Marañon 44–50
08028 Barcelona, Spain
e-mail: eva.miranda@upc.edu

Zdzisław Pogoda
Institute of Mathematics
Jagiellonian University
ul. prof. Stanisława Łojasiewicza
30-348 Kraków, Poland
e-mail: zdzislaw.pogoda@uj.edu.pl

Vladimir L. Popov
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina 8
119991 Moscow, Russia
e-mail: popovvl@mi.ras.ru

Themistocles M. Rassias
(Problem Corner)
Department of Mathematics
National Technical University of Athens, Zografou Campus
GR-15780 Athens, Greece
e-mail: trassias@math.ntua.gr

Volker R. Remmert
(History of Mathematics)
IZWT, Wuppertal University
D-42119 Wuppertal, Germany
e-mail: remmert@uni-wuppertal.de

Dierk Schleicher
Research I
Jacobs University Bremen
Postfach 750 561
28725 Bremen, Germany
dierk@jacobs-university.de

Olaf Teschke
(Zentralblatt Column)
FIZ Karlsruhe
Franklinstraße 11
10587 Berlin, Germany
e-mail: teschke@zentralblatt-math.org

Jaap Top
University of Groningen
Department of Mathematics
P.O. Box 407
9700 AK Groningen,
The Netherlands
e-mail: j.top@rug.nl

European Mathematical Society

Newsletter No. 95, March 2015

Editorial – *P. Exner* **3**
Farewells within the Editorial Board of the EMS Newsletter **4**
New Editors Appointed **4**
New Section of the Newsletter: YMCo – *J. Fresán & V. R. Kostic* .. **5**
New Members of the EC of the EMS..... **5**
EMS Paper on Open Access **7**
EMS Executive Committee Meeting in Barcelona – *S. Huggett* **8**
EU-MATHS-IN, Year 1 – *M. J. Esteban & Z. Strakoš* **10**
Joint AMS-EMS-SPM Meeting, 10–13 June 2015, Porto – *S. A. Lopes* **14**
SMAI Journal of Computational Mathematics – *D. N. Arnold & T. Goudon* **14**
The Pre-history of the European Mathematical Society – *Sir M. Atiyah*..... **17**
The *h*-Principle and Onsager's Conjecture – *C. De Lellis & L. Székelyhidi*..... **19**
Cannons at Sparrows – *G. M. Ziegler* **25**
Weierstrass and Uniform Approximation – *J. Cerdà* **33**
José Sebastião e Silva (1914–1972) – *J. Buescu et al.* **40**
Grothendieck and Algebraic Geometry – *L. Illusie & M. Raynaud*... **45**
Interview with Fields Medallist Artur Avila – *U. Persson* **48**
Solid Findings: Students' Over-reliance on Linearity – *Education Committee of the EMS* **51**
A Glimpse at the Reviewer Community – *O. Teschke* **55**
Book Reviews **57**
Problem Corner – *Th. M. Rassias* **61**

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ETH-Zentrum SEW A27
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EMS Executive Committee

President

Prof. Pavel Exner
(2015–2018)
Doppler Institute
Czech Technical University
Břehová 7
CZ-11519 Prague 1
Czech Republic
e-mail:ems@ujf.cas.cz

Vice-Presidents

Prof. Franco Brezzi
(2013–2016)
Istituto di Matematica Applicata e Tecnologie Informatiche del C.N.R.
via Ferrata 3
I-27100 Pavia
Italy
e-mail: brezzi@imati.cnr.it

Dr. Martin Raussen
(2013–2016)
Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg Øst
Denmark
e-mail: raussen@math.aau.dk

Secretary

Prof. Sjoerd Verduyn Lunel
(2015–2018)
Department of Mathematics
Utrecht University
Budapestlaan 6
NL-3584 CD Utrecht
The Netherlands
e-mail: s.m.verduynlunel@uu.nl

Treasurer

Prof. Mats Gyllenberg
(2015–2018)
Department of Mathematics and Statistics
University of Helsinki
P.O. Box 68
FIN-00014 University of Helsinki
Finland
e-mail: mats.gyllenberg@helsinki.fi

Ordinary Members

Prof. Alice Fialowski
(2013–2016)
Institute of Mathematics
Eötvös Loránd University
Pázmány Péter sétány 1/C
H-1117 Budapest
Hungary
e-mail: fialowsk@cs.elte.hu

Prof. Gert-Martin Greuel
(2013–2016)
Department of Mathematics
University of Kaiserslautern
Erwin-Schroedinger Str.
D-67663 Kaiserslautern
Germany
e-mail: greuel@mathematik.uni-kl.de

Prof. Laurence Halpern
(2013–2016)
Laboratoire Analyse, Géométrie & Applications
UMR 7539 CNRS
Université Paris 13
F-93430 Villetaneuse
France
e-mail: halpern@math.univ-paris13.fr

Prof. Volker Mehrmann
(2011–2014)
Institut für Mathematik
TU Berlin MA 4–5
Strasse des 17. Juni 136
D-10623 Berlin
Germany
e-mail: mehrmann@math.TU-Berlin.DE

Prof. Armen Sergeev
(2013–2016)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow
Russia
e-mail: sergeev@mi.ras.ru

EMS Secretariat

Ms. Terhi Hautala
Department of Mathematics and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
FIN-00014 University of Helsinki
Finland
Tel: (+358)-9-191 51503
Fax: (+358)-9-191 51400
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dr. Richard H. Elwes
School of Mathematics
University of Leeds
Leeds, LS2 9JT
UK
e-mail: R.H.Elwes@leeds.ac.uk

EMS Agenda

2015

6–8 March
Executive Committee of the EMS
Prague, Czech Republic

21–22 March
Ethics Committee Meeting
Istanbul, Turkey

28–29 March
Meeting of Presidents
Innsbruck, Austria

10–11 April
Committee for Developing Countries Round Table & Meeting
Oslo, Norway

29–31 May
Committee for Raising Public Awareness
Lisbon, Portugal

31 August–4 September
Meeting of the Women in Mathematics Committee during the 17th EWM General Meeting
Cortona, Italy

EMS Scientific Events

2015

23–26 March
Komplex Analysis Weeklong School – KAWA6 (EMS Summer School)
Pisa, Italy
http://www.ub.edu/kawa6/

1–5 June
Géométrie Algébrique en Liberté – GAeL XXII (EMS Summer School)
Leuven, Belgium
http://gael-math.org/

10–13 June
AMS-EMS-PMS Congress
Porto, Portugal
http://aep-math2015.spm.pt/
EMS distinguished speaker: Sylvia Serfaty (Paris, France)

6–10 July
European Meeting of Statisticians,
Amsterdam, The Netherlands
http://www.ems2015.nl/
Bernoulli Society-EMS Joint Lecture: Gunnar Carlsson (Stanford, CA, USA)

31 August–4 September
17th EWM General Meeting
Cortona, Italy
http://www.europeanwomeninmaths.org/
EMS Lecturer: Nicole Tomczak-Jaegerman (Edmonton, Alberta, Canada)

Editorial

Pavel Exner (EMS President 2015–2018)

Dear members of the EMS,

Let me start by wishing you a Happy New Year marked with good health and humour, as well as inspired ideas. Opening the next four-year cycle in the life of our society, I have the opportunity to greet all of you as I assume the EMS Presidency. It was your decision, through the council delegates, to put me in this role and I feel a deep gratitude for your trust and, at the same time, a grave responsibility thinking of what the job requires.

The challenge comes mostly from the achievements of my predecessors, especially my immediate predecessor Marta Sanz-Solé, who led the EMS over the last four years with incredible skill and commitment. She knew the society mechanisms in minute detail and steered its development in a way which was both diplomatic and strong. Her name is associated with numerous achievements which the European mathematical community has appreciated and enjoyed. The standards she set will always be on my mind.

The EMS is slowly coming of age. In the autumn of this year, it will be 25 years since the meeting in Ma(n)dralin near Warsaw at which 28 national societies agreed, after difficult negotiations, to board a single ship. It proved to be a good decision. After a quarter of a century, the number of member societies have more than doubled, to say nothing of close to 3,000 individual members and numerous other constituents. Activities of the EMS cover almost every aspect of a mathematician's life.

We are going to celebrate the anniversary with a small meeting in Paris in October. Rather than praising past achievements, however, the meeting's main focus will be on challenges that mathematics will have to face in Europe and globally in the years to come. Recent progress has opened up applications of mathematics in new areas and has brought to light new objects to be studied, as well as, for example, new requirements from publishers of mathematical texts; all this has to be reflected.

There are also challenges that arise on shorter timescales. We all know the useful role played by European instruments for funding research, in particular the European Research Council and the Marie Curie-Sklodowska scheme. The new European Commission has decided to help the economy with an investment package subtracted from the existing European budget, in part from the Horizon 2020 programme. The need for research is thus quoted as a pretext for taxing this excellent research funding. The EMS has joined other European scientific and learned societies in pointing out the flawed logic of such reasoning.

Looking within the society, we see many interesting activities ahead. There will be two EMS Supported Schools and a Joint Mathematical Weekend, lectures by EMS Distinguished Speakers and a Bernoulli Society-EMS Joint Lecture, to name just a few events. On some occasions, we join forces with partner societies, an example being the AMS-EMS-SMP meeting in Porto. I am convinced that many EMS members will also attend the ICIAM in Beijing, which traditionally convenes in the year following the ICM.

Our own congress is not that long away either; just some 18 months separates us from the moment when it will open at the Technical University of Berlin. The preparations are gaining pace. The programme committee, headed by Tim Gowers, should soon present its deliberations and we are opening nomination calls for prizes to be awarded at the opening of the congress. As these prizes serve to distinguish the best and the brightest among the younger generation of mathematicians, I encourage you to pay attention and to think of who, in your view, deserves the accolade.

There are many other issues to address – development of digital mathematical libraries, the future of the EMS Publishing House, support to the EU-MATHS-IN initiative aiming at industrial applications of mathematics, and others – but an opening message like this one should be brief. I thus prefer to stop here and will return to those questions as we go.

When Marta Sanz-Solé wrote her first presidential message after taking over from her predecessor, she likened her feelings to those of a runner in a relay race after gripping the baton. I think it is a very fitting metaphor: you are fully aware that it was the effort of the previous runners that brought you here but you have to look ahead and run with all force without stumbling. So let us run.

Pavel Exner
EMS President

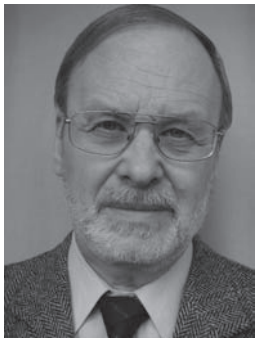
Farewells within the Editorial Board of the EMS Newsletter

In December 2014, the terms of office ended for **Mariolina Bartolini Bussi**, **Mădălina Păcurar** and **Ulf Persson**. We express our deep gratitude for all the work they have carried out with great enthusiasm and competence, and thank them for contributing to a friendly and productive atmosphere.
Three new members have rejoined the Editorial Board in January 2015. It is a pleasure to welcome Jean-Luc Dorier, Javier Fresán and Vladimir Popov, introduced below.

New Editors Appointed

Jean-Luc Dorier is a professor of mathematics didactics at Geneva University. His early research work was about the teaching and learning of linear algebra at university, which also led him to investigate several historical aspects, including Grassmann's *Ausdehnungslehre*. He wrote his *Habilitation* thesis on the interaction between the history and didactics of mathematics. He has taught mathematics to science and economics students and now teaches didactics of mathematics for both primary and secondary school teacher training programmes. He has research interests in mathematics education and the history of mathematics. Since January 2013, he has been an elected member of the Executive Committee of the *International Commission on Mathematical Instruction* (ICMI).

Javier Fresán wrote his thesis in arithmetic geometry at the University Paris 13 under the supervision of Christophe Soulé and Jörg Wildeshaus. After a year at the MPIM in Bonn, he is currently a post-doctoral fellow at the ETH in Zürich. His main research interests include periods, motives, special values of L-functions and the arithmetic of flat vector bundles. He is also an author of popular science books, including *The Folly of Reason and Amazing Algebra*, originally written in Spanish and translated into several languages.



Vladimir L. Popov is a leading research fellow at the Steklov Mathematical Institute, Russian Academy of Sciences. His main research interests are algebraic transformation groups, invariant theory, algebraic and Lie groups, automorphism groups of algebraic varieties and discrete reflection groups. He is Executive Managing Editor of *Transformation Groups*, Birkhäuser Boston (1996–present), and has been a member of the editorial boards of *Izvestiya: Mathematics* (2006–present) and *Mathematical Notes* (2003–present), Russian Academy of Sciences, the *Journal of Mathematical Sciences* (2001–present) and *Geometriae Dedicata* Kluwer (1989–1999). He is founder and Title Editor of the series *Invariant Theory and Algebraic Transformation Groups* of the *Encyclopaedia of Mathematical Sciences*, Springer (1998–present), and he is a fellow of the American Mathematical Society (since November 2012).

He was an invited speaker at the International Congress of Mathematicians, Berkeley, USA (1986), and a core member of the panel for Section 2, "Algebra", of the programme committee for the 2010 International Congress of Mathematicians (2008–2010). His webpage can be found at http://www.mathnet.ru/php/person.phtml?&personid=8935&option_lang=eng.

New Section of the Newsletter: YMCo

Javier Fresán (ETH Zürich, Switzerland) and Vladimir R. Kostic (University of Novi Sad, Serbia)

In the following issue of the EMS Newsletter, we are launching a new section – Young Mathematicians' Column (YMCo).

In YMCo, we will address different subjects that concern young mathematicians and young scientists in general. Its goal is to evolve through the years and offer perspectives that will attract a young audience to the Newsletter of the European Mathematical Society and encourage them to participate in EMS events, express their views and join discussion panels and round tables, be proactive in information sharing through EMS Newsletter Social Networks and become active participants in raising the impact of the EMS in the global scientific community.

This column is envisioned as a bridge between the particularities of different generations of mathematicians, so that the whole community may benefit from the complementarities that are to be discovered and integrated.

Periodically, among many other things, we will publish articles that:

- Discuss opportunities and difficulties that young mathematicians encounter in their early careers, both inside and outside academia.
- Disseminate information about prizes for young scientists in research and education.
- Discuss potentials and constraints on young people participating in the creation of the development strategies of science in Europe and worldwide.
- Collect memories and career advice from senior mathematicians that may help young people in their first steps in research, e.g. "How I proved my first theorem" or "Writing advice".
- Follow mathematical breakthroughs either by young people or in open-science networks.
- Promote interdisciplinary cooperation experienced through the life of young mathematicians.
- Promote various proactive groups within the European mathematical community.
- Raise the global awareness of the need for equal access to contemporary research and education for all genders.

Please feel free to contact us with your interests. We will be glad to reply and make this column up to date, informative and attractive for all our readers.

We are looking forward to our joint enterprise, in the name of the Editorial Board.

New Members of the EC of the EMS

Sjoerd Verduyn Lunel studied mathematics with physics at the University of Amsterdam and received a PhD from Leiden University in 1988. He is currently a professor of applied analysis at Utrecht University. He has held positions at Brown University, the Georgia Institute of Technology, the University of Amsterdam, Vrije Universiteit Amsterdam and Leiden University. He has been a visiting professor at the University of California at San Diego, the University of Colorado, the Georgia Institute of Technology, the University of Rome "Tor Vergata" and Rutgers University.

He was the Head of both the Mathematical Institute and the Leiden Institute of Advanced Computer Science at Leiden University from 2004 to 2007, and the Dean of the Faculty of Science at Leiden University from 2007 to 2012. He is currently the Scientific Director of the Mathematical Institute at Utrecht University and the Chair of the Board of the national platform for Dutch mathematics. He has been co-Editor-in-Chief of *Integral Equations and Operator Theory* (2000–2009) and is currently Associate Editor of *SIAM Journal on Mathematical Analysis* and of *Integral Equations and Operator Theory*.

His research interests are at the interface of analysis and dynamical systems theory. In his recent work he combines the theory of non-selfadjoint operators (in particular characteristic matrices, completeness, positivity and Wiener–Hopf factorisation) with a number of new techniques from analysis (in particular growth and regularity of subharmonic functions) and dynamical systems theory (exponential dichotomies and invariant manifolds). Applications include perturbation theory for differential delay equations and algorithms to compute the Hausdorff dimension of conformally self-similar invariant sets. He is co-author of two influential books on differential delay equations. Aside from this work, he is also interested in the development of algorithms for time series analysis using ideas and techniques from the theory of dynamical systems.

In 2012, he was elected as a member of the Royal Holland Society of Sciences and Humanities and, in 2014, he was appointed honorary member of the Indonesian Mathematical Society.

Mats Gyllenberg was born in Helsinki in 1955 and studied mathematics and microbiology at the Helsinki University of Technology, from where he received his doctorate in mathematics. After having held positions of acting associate professor at the Helsinki University of Technology and research fellow at the Academy of Finland, he was appointed a full professor of applied mathematics at Luleå University of Technology in 1989. From 1992 to 2004 he was a professor of applied mathematics at the University of Turku. Since 2004, he has been a professor of applied mathematics at the University of Helsinki, where he has been the Chairman of the Department of Mathematics and Statistics since 2008.

He was a visiting researcher at the Mathematisch Centrum in Amsterdam, the Netherlands, in the academic year 1984–1985. He has held visiting professorships at the following universities and institutes: Vanderbilt University, Nashville, Tennessee (1985–1986), National Center for Ecological Analysis and Synthesis, Santa Barbara, California (1996), University of Utrecht, the Netherlands (1997, 2006, 2007) and Chalmers University of Technology (1998). In 2006, he held the F.C. Donders Visiting Chair of Mathematics at the University of Utrecht.

His research interests include stochastic processes and infinite dimensional dynamical systems arising in the study of delay equations, Volterra integral equations and partial differential equations and their applications to biology and medicine. He has published more than 200 papers and three books.

He is the Editor-in-Chief of the *Journal of Mathematical Biology* and of *Differential Equations and Applications*.

He has been the Chairman of the Standing Committee for Physical and Engineering Sciences of the European Science Foundation since 2009 and a member of the Council and Executive Committee of the International Institute for Applied Systems Analysis in Laxenburg, Austria, since 2012. He was a member of the Mathematics Panel of the European Research Council (ERC) from 2007 to 2012.

He is an elected member of the following learned societies: the Swedish Academy of Engineering Sciences in Finland (1996), the Finnish Academy of Science and Letters (2008), the Finnish Society of Sciences and Letters (2009) and the European Academy of Sciences (2010).

Reaction to the Juncker Plan

The EMS Executive Committee

As mentioned at another place, the new European Commission decided to start its term by the project called "Investment Plan for Europe", injecting some 16 billion euro into the European economy with the idea to revive it and the hope for a huge multiplication effect. The goal is noble, of course, but the question is where the money will be taken from. It appears that the Horizon 2020 program should contribute 2.7 billion, and even two of its chapters considered as front drivers of excellent research in Europe, the European Research Council and the Marie Sklodowska-Curie Program are supposed to be taxed by 221 and 100 million, respectively. Other chapters should contribute also with one exception – strangely enough the chapter called "Access to Risk Finance" is left untouched. Details of the plan can be found at ec.europa.eu/priorities/jobs-growth-investment/plan/documents/index_en.htm

Even if those sums are a small part of the whole package, the effect would be significant. For instance, the ERC would in this way lose some 150 grants which amounts to roughly one half of a full call; note that the money is taken for the seven-year budget but the cuts will be concentrated to a much short period so they will hurt indeed. Aware of the implications, various European organizations raise their voices in protest to this decision, among others Academia Europaea, League of European Research Universities (LERU), or EuroScience, to name just a few. Recently the Initiative for Science in Europe (ISE), of which the EMS is a member, suggested to European researchers to tell their opinion to their elected representatives. The plan cannot take effect without being approved by the European Parliament whose members are there to represent interests of their constituencies. The campaign is decribed at

http://www.no-cuts-on-research.eu/emailcampaign/

and we recommend it to the attention of the EMS members.

EMS Paper on Open Access

The EMS Executive Committee

In the Editorial of the EMS Newsletter published in June 2013, the following statement was made:

> *"The EMS endorses the general principle of allowing free reading access to scientific results and declares that in all circumstances, the publishing of an article should remain independent of the economic situation of its authors. We therefore do not support any publishing models where the author is required to pay charges (APC)".*

Establishing sound and sustainable procedures for OA, if possible in co-operation with other learned societies and institutions, is among the priorities of the Society.

The EMS endorses the goals of the ICSU document *Open access to scientific data and literature and the assessment of research by metrics*[1] that the scientific record should be:

- free of financial barriers for any researcher to contribute to and for any user to access immediately on publication;
- made available without restriction on reuse for any purpose, subject to proper attribution;
- quality-assured and published in a timely manner;
- archived and made available in perpetuity.

The mechanisms for financing publications should be chosen and adjusted taking into account the scientific judgement of the end users, as opposed to a "supply side economy" of knowledge that is extremely costly.

A clear distinction should be made between open *diffusion* of knowledge and ideas – as offered for example by arXiv – and *publications*. Publications should be validated by a refereeing process of proper quality, and their long-term availability assured. This was made more relevant than ever by the explosion of electronic OA publications. Many of these publications claim to be peer reviewed, but a lack of standards and a large number of papers has led instead to a vast grey area between diffusion and publications.

There is a necessity to establish a code of good practice in publication encompassing all its aspects, including the peer reviewing system, contribution to the creation of a searchable scientific corpus, deposit in an open repository after a reasonable embargo period, and long-term accessibility.

Scientific libraries in co-operation with user committees have played an important role in evaluating, organizing, and preserving scientific documents. This must be maintained. Moreover, scientists should use their expertise to help their libraries to adapt to the new environment provided by the new digital technologies.

For non-commercial purposes, mathematical papers and data, including metadata, should be freely accessible.

The EMS will strive to develop the following.

A charter of good practice in publication

The *EMS Code of Practice* prepared by the Ethics Committee and approved by the Executive Committee in 2012 emphasises ethical aspects of publication, dissemination, and assessment of mathematics. These and other crucial aspects of publication, like the contribution to a searchable scientific corpus, the depositing of papers in an OA archive after a reasonable embargo period, the guarantee of free long-term access to published scientific articles, should comprise a *Charter of Good Practice in Publication*. It is recommended to have this *Charter* elaborated in consensus with other learned societies, and adhered to by all mathematicians, editors, and publishers of mathematics.

The EMS will recommend its members to pay close attention to the ways in which particular publishers follow good practices, and to take responsibility in their publication, editorial, and evaluation activities for avoiding publications that fail to follow good practices. This is to help stabilize the publishing system and to discriminate good publishers from predatory ones.

Public funding for scientific documentation

By exerting its influence in Europe, the EMS will encourage European and national research funding agencies to become concerned with the future of scientific documentation and with the control of its costs. It is an obligation for public research institutions to fulfil their responsibility for the organized preservation of knowledge. Libraries will not be able to adapt to their new roles without specific funds. The costs should be taken into account in the research budgets.

The EMS joins ICSU in the recommendation that the terms of contracts governing the purchase of scientific periodicals and databases by libraries serving universities and research establishments should be publicly accessible.

Databases and digital libraries

Databases and data mining are fundamental for research activities. The comprehensive database in the mathematical sciences, zbMATH, edited by the EMS, FIZ Karlsruhe, and the Heidelberg Academy of Sciences, will enlarge the scope of its activities to become a stronger and scientifically more reliable search and data-mining instrument for all types of mathematical documents.

[1] http://www.icsu.org/general-assembly/news/ICSU%20Report%20on%20Open%20Access.pdf

Digital libraries provide access to digital documents. The EMS has been supporting the development of the European Digital Mathematics Library (EuDML)—an access platform for digital mathematical content hosted by different organizations across Europe. There is a necessity to continue its development as an open library and archive, and to make it part of the world project *Global Library for Mathematics Research.*

The EMS makes the following recommendations.

Libraries in the new era
With the new digital technologies, mathematical and all scientific libraries have become a very complex environment, and the notion of document much broader (it now includes software, videos, blogs, and so on). Nevertheless, the crucial role of libraries and librarians remains essentially the same, although the tools needed to carry out the tasks are much more numerous. Helping the adaptation to this new sophisticated environment will benefit the users and contribute to the advancement of research. The EMS strongly recommends its members to play an active and co-operative role in this important task.

Quality indicators
Decisions on subscriptions to journals should be guided by the services they render, the quality of the reviewing process, the editing, the contribution to the advancement of mathematics, deposition in free access archives in a way compatible with the currently available search engines, the guarantee of long-term preservation, etc. These are more meaningful qualities than impact factors.

Endorsement by scientific users and libraries of these quality indicators would provide support for the best journals, help to identify quality in the jungle of new publications, and assist in the careful selection of subscriptions to bundles.

It should be in the best interest of universities and other research institutions to adopt these guidelines for an optimal use of funds devoted to libraries.

San Francisco Declaration on Research Assessment
The EMS endorses the San Francisco Declaration on Research Assessment (DORA), which recognizes the danger in the use of impact factors in evaluation. Corporate and individual members of the EMS are strongly encouraged to sign the declaration.

December 2014

This paper is based on a document prepared by the EMS Publications Committee available at http://www.euro-math-soc.eu/system/files/uploads/EMS.Draft_.v10.pdf

EMS Executive Committee Meeting in Barcelona on the 21st and 22nd of November 2014

Stephen Huggett (University of Plymouth, UK)

Preliminaries
On Friday the meeting was hosted by the University of Barcelona, and Carme Cascante, the Dean of the Faculty of Mathematics of the University of Barcelona, welcomed the Executive Committee. On Saturday the meeting was hosted by the Catalan Mathematical Society, and the Executive Committee was welcomed by Joandomènec Ros, the President of the Institute of Catalan Studies, and by Joan Solà-Morales, the President of the Catalan Mathematical Society.

Treasurer's Report
The Treasurer presented a report on income and expenditure. In the ensuing discussion, it was agreed that in future the contribution from the Department of Mathematics at the University of Helsinki would be made explicit. It was also agreed that for the moment the Society needed to retain both routes to individual membership: through a member Society and directly online.

The President welcomed the significant increase in the expenditure on scientific projects, and noted that in order to give appropriate support to all our summer schools it is hoped to raise external funds, which may be easier now that we have started the programme.

Membership
The applications for Institutional Membership from the Basque Centre for Applied Mathematics and the Department of Mathematics of Stockholm University were approved.

The Executive Committee approved the list of 74 new individual members.

EMS on the Internet

Martin Raussen introduced the new web site, and described the process by which logged-in members can upload material to the site, noting that it is moderated before being published.

It was agreed that an editorial board is needed for the web site, including the oversight of our presence on social media.

Scientific Meetings and Activities

The Executive Committee discussed the report on the 7th European Congress of Mathematics from Volker Mehrmann, noting that pre-registration is now open. Then the President presented the report from Tim Gowers on the work of the 7ECM Scientific Committee.

The President reported that the EMS Summer Schools had all been very good, as had the joint EMS-IAMP school on general relativity. The next such joint school would be held in 2016.

Amen Sergeev reported on the Caucasian Mathematics Conference, starting by describing the work of the steering committee. The planning for the conference was made extremely difficult by the failure of the Shota Rustaveli Science Foundation to award a grant, but at the very last minute a grant was obtained from the Georgian Ministry of Education and Science. The attendance at the Conference, especially by participants from Iran, was badly affected by new visa regulations, announced by the Georgian Government just ten days before the Conference opened. The scientific level of the plenary speakers was very high, and the main aim, to start to build horizontal connections in the region, had been achieved. The next Caucasian Mathematics Conference would be held in Turkey in 2016.

Several future events were discussed, such as the Joint Mathematical Weekend with the London Mathematical Society to be held in Birmingham in September 2015.

The Treasurer reported on a discussion with the President of UMALCA (the Unión Matemática de América Latina y el Caribe) in which the idea of the EMS sending a lecturer to their next meeting – in Bogotá in 2015 – was suggested. The Executive Committee agreed with this proposal.

Society Meetings

It was noted that the Meeting of Presidents would be held in Innsbruck on the 28th and 29th of March 2015.

A meeting to celebrate the 25th anniversary of the EMS would be held at the Institut Henri Poincaré in Paris, on the 22nd of October 2015. A small committee for planning this event would be appointed.

Publicity Officer

The report from the Publicity Officer was received. The Executive Committee agreed to thank Dmitry Feichtner-Kozlov for his work, and the University of Bremen for its support. It was then agreed to appoint Richard Elwes as the Publicity Officer for 2015–2018.

Publishing

The President introduced a discussion on the future direction of the EMS publishing house by expressing her concern that the two organisations, the EMS and its publishing house, were not working closely enough together. In the longer term, she would like to see an agreement whereby profits from the publishing house came to the EMS. In the shorter term, the EMS should be providing more scientific direction to the publishing house.

Relations with Funding Organisations and Political Bodies

The President reported on a workshop in Brussels on *Big Data and High Performance Computing* which had been organised by DG Connect following an online consultation to which the EMS had been able to respond very effectively, making it clear that mathematics was crucial to this question. In the workshop itself the EMS participants were able to have a significant and coordinated influence. One outcome will be calls in 2016–2017 specifically for mathematics.

Pavel Exner reported on the budget squeeze on Horizon 2010 and on the ERC in particular, coming from the so-called Juncker plan, and on the challenges arising from bringing all the Horizon applications to a common platform. He also mentioned the forthcoming renewal of the ERC Scientific Council. Overall, things are running well, but the philosophy of the ERC still very much needs strong support from the community.

Closing

The President expressed the gratitude of the whole Executive Committee to the University of Barcelona, the Institute of Catalan Studies, and the Catalan Mathematical Society for their excellent hospitality.

Martin Raussen proposed a vote of thanks to the outgoing President, Secretary, and Treasurer.

EU-MATHS-IN, Year 1

Maria J. Esteban (CEREMADE, Paris, France) and Zdeněk Strakoš (Charles University, Prague, Czech Republic)

A little more than a year ago, with the ECMI and the EMS as promoting members, the European EU-MATHS-IN initiative was launched to support applied and industrial mathematics in Europe. EU-MATHS-IN is a network of national networks that represent the community in their respective countries. In 2013, at its creation, there were six national network members. Currently 13 countries are already on board: IMNA (Austria), EU-MATHS-IN.cz (Czech Republic), AMIES (France), KoMSO (Germany), HSNMII (Hungary), MACSI (Ireland), Sportello Matematico (Italy), NNMII (Norway), PL-MATHS-IN (Poland), Math-in (Spain), EU-MATHS-IN.se (Sweden), PWN (The Netherlands) and the Smith Institute (UK). Finland and Portugal are getting ready to adhere soon and Bulgaria is working on it. This is an incredible growth rate!

The actions taken by EU-MATHS-IN over the first year of its existence should strengthen the position of the whole mathematics community in Europe and we think that sharing them with all mathematicians will serve as an inspiration for similar initiatives.

One of the most important activities of EU-MATHS-IN over the first year of its existence has been the strengthening of relations with various bodies of the EU in Brussels. During Spring 2014, EU-MATHS-IN, together with the ECMI and the EMS, launched a campaign to push 'Modeling, Simulation and Optimization' as a future Key Enabling Technology (KET). A comprehensive position paper about this issue can be found at http://www.eu-maths-in.eu/index.php?page=generalReports. Later, in June 2014, EU-MATHS-IN launched a Proactive Proposal for establishing 'Mathematical Modeling, Simulation and Optimization' as a Future Emerging Technology (FET) within the framework of the European programme Horizon 2020.

As a result of the Proactive FET proposal, the EU Directorate General for Communications Networks, Content and Technology (DG CNECT) decided to launch an online consultation on 'Mathematics and Digital Science' (mentioning High Performance Computing (HPC) in the FET Proactive text was the reason for DG CNECT to couple the initiative to 'Digital Science'). This triggered many likes and many comments, showing the strength of the 'network of networks' concept of EU-MATHS-IN. This also led to DG CNECT organising a workshop in Brussels, on 6 November, on the topic 'Mathematics and Digital Sciences'. The EMS and the EuDML were also heavily involved in this workshop. The outcome of all these actions is a document that synthesises the contributions of the workshop, which will serve as a source of inspiration for the writing of the Horizon 2020 Work Programme 2016-2017 of DG CNECT: http://www.eu-maths-in.eu/download/generalReports/Mathematics_for_the_digital_science_report_Brussels_Nov6.2014.pdf.

Moreover, after the publication of this report, a new meeting took place in December between DG CNECT and EU-MATHS-IN to discuss how the mathematical community could enter into EU programmes designed by this DG.

On the other hand, as a result of the Spring KET campaign, EU-MATHS-IN was invited to a meeting with the EU Directorate General for Research and Innovation (DG RTD). At this meeting, which took place on 25 September, the role of mathematical sciences within the Horizon 2020 Work Programme 2016-2017 was discussed at length.

It is clear that we have opened several new doors in Brussels this year. These actions could prove to be fruitful for the future funding of mathematical projects by the EU but a lot of further effort and patience is needed, since the mathematical community is significantly behind in this respect. EU-MATHS-IN has coordinated a COST proposal 'Modeling, Simulation, Optimization and Control of Large Infrastructure Networks' that has not been accepted but there have been two other COST actions launched by the mathematical community that have been funded!

Following one of the main goals of EU-MATHS-IN, an e-infrastructure proposal has been prepared and recently submitted within the EC call EINFRA-9-2015. This proposal contains work packages corresponding to the aims and projects of EU-MATHS-IN, as well as some work packages devoted to digital mathematical libraries, led by the consortium EuDML. The preparation of this proposal has represented a huge investment of time and energy. Even if several European officials have warned us that such a proposal is not likely to be accepted the first time it is submitted, we remain optimistic and also ready to take into account the possible advice given by potential referees to improve it for another submission.

Another important project of EU-MATH-IN has been the launch of the European job portal for jobs in companies or academia but related to industrial contracts. Due to their past experience, this portal has been set up by AMIES, the French network within EU-MATHS-IN. Within the e-infrastructure, this project could play a very important role for the visibility of EU-MATHS-IN among companies all over Europe.

Other important actions of this first year are:

- A two-day meeting with the President and the two Vice-Presidents of SIAM. The goal was to exchange information on actions related to mathematics and industry both in the US and in Europe, to discuss existing documents and to define a strategy for further collaboration. As a result, there have already been some actions (see below) and there is a project for a SIAM annual conference, organised in Europe jointly with EU-MATHS-IN or one of its networks.

- Publication of an article in SIAM News, December 2013.
- Presentation of EU-MATHS-IN at the European Mathematics Representatives Meeting (EMRM) (Helsinki, 9 May 2014).
- Presentation of EU-MATHS-IN at the ICIAM Council Meeting (Columbus, 17 May 2014).
- Participation in the ECMI Conference (Taormina, 11 June 2014).
- Panel discussion at the SIAM Annual Meeting, July 2014.
- Presentation of EU-MATHS-IN at the Meeting of the Portuguese Mathematical Society (Braga, 26 September 2014).
- Presentation of EU-MATHS-IN at the Meeting of the EMS-AMC (Applied Mathematics Committee of the EMS) (London, 24 October 2014).
- Discussion with the European Network for Business and Industrial Statistics (ENBIS, www.enbis.org) about possible collaborations. ENBIS has expressed an interest in joining EU-MATHS-IN but this is not possible unless there is a change of statutes. Cross-participation in council meetings and discussion of joint initiatives will take place.

As a conclusion, EU-MATHS-IN and its national networks have been very busy in the organisation of the European network but, at the same time, there have already been a large number of concrete actions and the level of coordination shown by the different structures all over Europe has been incredibly quick and effective. The rapid increase in the number of countries that are part of our network is a very encouraging factor.

The contacts of EU-MATHS-IN with several European Commission structures in Brussels (DG RTD and DG CNECT) have been important but the momentum has to be maintained at a high level if we want to succeed in giving mathematics more opportunities in Brussels.

We should also be present and active in the emerging discussions on the role of mathematics in applications and actively explain our views. The inspirational article titled "Is Big Data Enough? A Reflection on the Changing Role of Mathematics in Applications", by Napoletani, Panza and Struppa, which appeared in the May 2014 issue of the Notices of the AMS, shows that the ideas presented by EU-MATHS-IN on various occasions have a very sound and resonating background. In particular, as convincingly argued many times in the article mentioned above, HPC and Big Data – the only mathematical items that are nowadays present in European programmes – require mathematics not only in the form of particular tools for solving particular problems but, more substantially, as an integrated methodology to be developed in order to enable understanding of the phenomena.

Report on the Meeting of the Education Committee (Prague, 9th February 2015)

Günter Törner (Universität Duisburg-Essen, Germany)

At the 9th Congress of European Research in Mathematics Education (CERME) (Prague), the Education Committee met and was honoured with the presence of the new EMS President Pavel Exner. It was an encouraging discussion about the plan for future work within the EMS. In addition, the committee's chair Günter Törner and the President of ERME Viviane Durand Guerrier (France) organised a public address and led an intensive discussion with many congress participants about future joint projects.

Fermat Prize

The new edition of the **Fermat Prize for Mathematics Research** is released in March 2015; the call for nominations of candidates will be open until the 30th of June 2015 and the results will be announced in December 2015.
The Fermat Prize rewards the research work of one or more mathematicians in fields where the contributions of Pierre de Fermat have been decisive: Statements of Variational Principles, or more generally partial differential equations; Foundations of Probability and Analytical Geometry; Number Theory.
More informations about the Fermat Prize, in particular about the applications, are available at
http://www.math.univ-toulouse.fr/FermatPrize

Call for Nominations of Candidates for Ten EMS Prizes

Principal Guidelines
Any European mathematician who has not reached his/her 35th birthday on July 15, 2016, and who has not previously received the prize, is eligible for an EMS Prize at 7ecm. Up to ten prizes will be awarded. The maximum age may be increased by up to three years in the case of an individual with a broken career pattern. Mathematicians are defined to be European if they are of European nationality or their normal place of work is within Europe. Europe is defined to be the union of any country or part of a country which is geographically within Europe or that has a corporate member of the EMS based in that country. Prizes are to be awarded for work accepted for publication before October 31, 2015.

Nominations for the Award
The Prize Committee is responsible for the evaluation of nominations. Nominations can be made by anyone, including members of the Prize Committee and candidates themselves. It is the responsibility of the nominator to provide all relevant information to the Prize Committee, including a résumé and documentation. The nomination for each award must be accompanied by a written justification and a citation of about 100 words that can be read at the award ceremony. The prizes cannot be shared.

Description of the Award
The Foundation Compositio Mathematica has kindly offered to sponsor a substantial part of the prize money.

Award Presentation
The prizes will be presented at the Seventh European Congress of Mathematics in Berlin, July 18–22, 2016, by the President of the European Mathematical Society. The recipients will be invited to present their work at the congress.

Prize Fund
The money for the Prize Fund is offered by the *Foundation Compositio Mathematica*.

Deadline for Submission
Nominations for the prize must reach the chairman of the Prize Committee at the address given below, not later than November 1, 2015:

Professor Björn Engquist
The Institute for Computational Engineering and Sciences
The University of Texas at Austin
engquist@ices.utexas.edu

Call for Nominations of Candidates for The Felix Klein Prize

Background
Nowadays, mathematics often plays the decisive role in finding solutions to numerous technical, economical and organizational problems. In order to encourage such solutions and to reward exceptional research in the area of applied mathematics the EMS decided, in October 1999, to establish the *Felix Klein Prize*.The mathematician Felix Klein (1849–1925) is generally acknowledged as a pioneer with regard to the close connection between mathematics and applications which lead to solutions to technical problems.

Principal Guidelines
The Prize is to be awarded to a young scientist or a small group of young scientists (normally under the age of 38) for using sophisticated methods to give an outstanding solution, which meets with the complete satisfaction of industry, to a concrete and difficult industrial problem.

Nominations for the Award
The Prize Committee is responsible for solicitation and the evaluation of nominations. Nominations can be made by anyone, including members of the Prize Committee and candidates themselves. It is the responsibility of the nominator to provide all relevant information to the Prize Committee, including a résumé and documentation of the benefit to industry and the mathematical method used. The nomination for the award must be accompanied by a written justification and a citation of about 100 words that can be read at the award date. The prize is awarded to a single person or to a small group and cannot be split.

Description of the Award
The award comprises a certificate including the citation and a cash prize of 5000 €.

Award Presentation
The Prize will be presented at the Seventh European Congress of Mathematics in Berlin, July 18–22, 2016, by a representative of the endowing Fraunhofer Institute for Industrial Mathematics in Kaiserslautern or by the President of the European Mathematical Society. The recipient will be invited to present his or her work at the congress.

Prize Fund
The money for the Prize fund is offered by the *Fraunho fer Institute for Industrial Mathematics* in Kaiserslautern.

Deadline for Submission
Nominations for the prize should be addressed to the chairman of the Prize Committee, Professor Mario Primicerio (University of Florence). The nomination letter must reach the EMS office at the following address, not later than December 31, 2015:

EMS Secretariat
Ms. Elvira Hyvönen
Department of Mathematics & Statistics
P.O.Box 68 (Gustaf Hällströmink. 2b)
00014 University of Helsinki
Finland

Prize Committee Chair Address
Prof. Mario Primicerio
Dipartimento di Matematica "Ulisse Dini"
Università degli Studi di Firenze
Viale Morgagni 67/A
50134 Firenze, Italy
Tel: +39 055 2751439
Home: +39 055 4223458
Cell: +39 349 4901708
Fax: +39 055 2751452
E-mail: mario.primicerio@math.unifi.it

Call for Nominations of Candidates for The Otto Neugebauer Prize for the History of Mathematics

Principal Guidelines
The Prize is to be awarded for highly original and influential work in the field of history of mathematics that enhances our understanding of either the development of mathematics or a particular mathematical subject in any period and in any geographical region. The prize may be shared by two or more researchers if the work justifying it is the fruit of collaboration between them. For the purposes of the prize, history of mathematics is to be understood in a very broad sense. It reaches from the study of mathematics in ancient civilisations to the development of modern branches of mathematical research, and it embraces mathematics wherever it has been studied in the world. In terms of the Mathematics Subject Classification it covers the whole spectrum of item 01Axx (History of mathematics and mathematicians). Similarly, there are no geographical restrictions on the origin or place of work of the prize recipient. All methodological approaches to the subject are acceptable.

Nominations for the Award
The right to nominate one or several laureates is open to anyone. Nominations are confidential; a nomination should not be made known to the nominee(s). Self-nominations are not acceptable. It is the responsibility of the nominator to provide all relevant information to the Prize Committee, including a CV and a description of the candidate's work motivating the nomination, together with names of specialists who may be contacted.

Description of the Award
The award comprises a certificate including the citation and a cash prize of 5000 €.

Award Presentation
The prizes will be presented at the Seventh European Congress of Mathematics in Berlin, July 18–22, 2016, by the President of the European Mathematical Society. The recipients will be invited to present their work at the congress.

Prize Fund
The money for the Prize Fund is offered by *Springer Verlag*.

Deadline for Submission
Nominations for the prize should be addressed to the chairman of the Prize Committee, Professor Jesper Lützen (Copenhagen University). The nomination letter must reach the EMS office at the address given below, not later than December 31, 2015:

EMS Secretariat
Ms. Elvira Hyvönen
Department of Mathematics & Statistics
P.O.Box 68 (Gustaf Hällströmink. 2b)
00014 University of Helsinki
Finland

Joint AMS-EMS-SPM Meeting 10–13 June 2015, Porto Registration is Open

Samuel A. Lopes (University of Porto, Portugal)

The Joint International Meeting of the American, European and Portuguese Mathematical Societies will be held 10–13 June 2015 in the city of Porto, the 2014 winner of the award for the best destination in Europe. There will be nine invited plenary talks and 53 high-level special sessions focusing on the most recent developments in their fields, as well as a contributed paper session.

Several social events are planned, including a reception on 9 June and an evening public lecture by Marcus du Sautoy followed by a concert on 10 June and a conference dinner on 11 June. Detailed information can be found on the meeting website

http://aep-math2015.spm.pt.

The organising committee is pleased to announce that registration for the conference is open and that there are reduced fees for students and members of any of the organising societies. The special early-bird registration period ends on 30 April. Please go to

http://aep-math2015.spm.pt/Fees

for more details and registration instructions.

The organisers hope to see you in Porto for this exciting scientific event!

SMAI Journal of Computational Mathematics

Doug N. Arnold (University of Minnesota, Minneapolis, USA) and Thierry Goudon (INRIA, Sophia Antipolis, France)

Widely accessible, carefully peer-reviewed scientific literature is truly important. It is crucial to effective research and hence has significant impact upon the world's health, security and prosperity. However, the high cost of many journals blocks access for many researchers and institutions, and places an unsustainable drain on the resources of others. Addressing this issue, the Société de Mathématiques Appliquées et Industrielles (SMAI) – the French professional society for applied and industrial mathematics – has committed to the founding of a new journal of computational mathematics: *SMAI Journal of Computational Mathematics* (SMAI-JCM), which will be freely accessible to all and will not require the payment of fees for publication.

The journal, which has just commenced operations and is reviewing its first submissions, intends to publish high quality research articles on the design and analysis of algorithms for computing the numerical solution of mathematical problems arising in applications. Such mathematical problems may be continuous or discrete, deterministic or stochastic. Relevant applications span sciences, social sciences, engineering and technology. SMAI-JCM, reflecting the broad interests of a strong and diverse international editorial board, takes a broad view of computational mathematics, ranging from the more analytical (numerical analysis) to the more applied (scientific computing and computational science). In particular, the journal welcomes submissions addressing:

- Computational linear and nonlinear algebra.
- Numerical solution of ordinary and partial differential equations.
- Discrete and continuous optimisation and control.
- Computational geometry and topology.

- Image and signal processing.
- Processing of large data sets.
- Numerical aspects of probability and statistics; assessment of uncertainties in computational simulations.
- Computational issues arising in the simulation of physical or biological phenomena, engineering, the social sciences or other applications.
- Computational issues arising from new computer technologies.
- Description, construction and review of test cases and benchmarks.

As this list indicates, the editorial board recognises that excellence in computational mathematics arises from a broad spectrum of researchers and viewpoints, and encourages submissions of different sorts, with varying balance between computational results and theoretical analysis. Typically, the strongest submissions are expected to involve both aspects. The journal will also provide for the publication of supplementary material, such as computer codes and animations.

Peer review will be carried out at SMAI-JCM, just as in top traditional journals, and the journal will strive to maintain the highest ethical standards and to employ the best practices of modern, scholarly journal publication. However, the journal's business model is a radical departure from current practice. All papers accepted by SMAI-JCM will be electronically published in full open access, downloadable by anyone, without delay and in perpetuity. Publication in SMAI-JCM is also entirely free to authors, with the only barrier being scientific quality as determined by careful peer review. Of course, the publication of a high quality journal does incur costs, in addition to the freely provided efforts of authors, editors and referees. For SMAI-JCM, these financial costs are directly borne by the SMAI and other sponsoring organisations. We believe that this approach is the most promising way of achieving the goal of universal access to the scientific literature and we hope that a successful SMAI-JCM will not only improve the publishing of computational mathematics but serve as a model for other journals.

Context for the new journal can be found in a recent report[1] by the ICSU (the International Council for Science), whose members are primarily scientific unions, such as the International Mathematical Union and national academies of science. The report advocated the following goals, stating: "The scientific record should be:

- free of financial barriers for any researcher to contribute to;
- free of financial barriers for any user to access immediately on publication;
- made available without restriction on reuse for any purpose, subject to proper attribution;
- quality-assured and published in a timely manner; and
- archived and made available in perpetuity."

Unfortunately, these goals are far from being realised. In the area of computational mathematics, for example, a well-known computational physics journal charges annual subscription fees that vary between $6,652 and $11,396 for online, institutional access, which is much more than many institutions can afford.[2] Numerous other journals also charge very steep fees. Despite the massive revenues generated for the publisher by these fees, the articles published are not "free of financial barriers for any user to access immediately on publication" but only freely available to users from subscriber institutions. Authors wishing to have their papers placed in open access are required to pay an additional fee of $2,200.[3]

After studying the situation, the ICSU report concludes that the resources used to support scientific publication are sufficient to bring about a scientific literature as described above: free of financial barriers to access or contribution, while maintaining quality peer review and the best practices in publishing. The obstacle to such a system comes not from the available resources but rather from the current business models predominant in scholarly publishing. If these models are to change, it will surely have to be researchers themselves – the people who provide the content for the journals and carry out the key editorial and refereeing roles – to bring this about. Similar conclusions have been made in other reports. An October 2014[4] report of the French Academy of Sciences called on scientists to "regain control of costs for activities that relate to dissemination of scientific information", while reaffirming "the primary need for peer–reviewing of articles before publication by academic research scientists" and the importance of "participation of academics in the final approval decisions".

SMAI-JCM is responding to these calls, offering a model of journal publication which, if widely deployed, can make these goals a reality. Our success in this depends crucially on the acceptance and support of SMAI-JCM by the community. We very much encourage the submission of strong papers in computational mathematics to the journal. Please visit the journal at

https://ojs.math.cnrs.fr/index.php/SMAI-JCM

and help us take a step towards quality, accessible, ethical publishing in mathematics.

[1] http://www.icsu.org/general-assembly/news/ICSU%20Report%20on%20Open%20Access.pdf

[2] http://store.elsevier.com/product.jsp?issn=00219991

[3] http://www.elsevier.com/journals/journal-of-computational-physics/0021-9991/guide-for-authors#13510

[4] http://www.academie-sciences.fr/presse/communique/rads241014.pdf

The Gold Medal "Guido Stampacchia" Prize

Antonio Maugeri (University of Catania, Italy)

The Gold Medal "Guido Stampacchia" Prize is promoted by the International School of Mathematics "Guido Stampacchia" in collaboration with the Unione Matematica Italiana. It is assigned every three years to mathematicians not older than 35 who have made an outstanding contribution to variational analysis, the field in which Guido Stampacchia produced celebrated results.

The prize is assigned by an international committee and the winner is awarded at the international workshop "Variational Analysis and Applications" (varana.org), which takes place in Erice at the International School Guido Stampacchia.

The first winner was *Tristan Rivière* (ETH Zurich) in 2003, the second was *Giuseppe Mingione* (University of Parma) in 2006, the third was *Camillo De Lellis* (University of Zurich) in 2009 and the fourth was *Ovidiu Savin* (Columbia University) in 2012.

Fifth edition of the Gold Medal «Guido Stampacchia» Prize

The Unione Matematica Italiana announces the fifth international competition for assigning a gold medal to a researcher who is not older than 35 years on December 2015 and who has done meaningful research in the field of variational analysis and its applications. Nominations must be sent, before 31 March 2015, to:

Commissione Premio Stampacchia, Unione Matematica Italiana, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40126 BOLOGNA (ITALY)

An international committee will evaluate the nominations and assign the medal, which will be given on 29 August 2015 at the beginning of the International Conference "Variational Analysis and Applications", to be held in Erice (Sicily) at the "E. Majorana" Foundation, 28 August – 5 September 2015 (see varana.org).

The 25th First Years of the EMS Through Your Eyes

Dear Readers,

The EMS will celebrate its 25th anniversary in October 2015. We would like to retrace these years through your pictures and publish them, mainly on the web.
If you want to share your pictures **of any EMS related event**, please send them to the Editorial Board using the form at

http://divizio.perso.math.cnrs.fr/EMSnewsletter/

or by email to the address euro.math.soc.newsletter@gmail.com. Please specify who has taken the picture, where and when it was taken and please confirm that you have the rights to it.

The Pre-history of the European Mathematical Society

Sir Michael Atiyah (University of Edinburgh, UK)

This year, the EMS celebrates its 25th anniversary and I hasten to add my congratulations. From a modest beginning, it has expanded in many directions, including its own publications and the European Mathematical Congress. It is now a major player on the academic scene and is active in many countries.

Past presidents will recount the history during their terms of office and I have been asked to talk about the long gestation period before the foundation of the EMS, in which I was heavily involved. Unfortunately, written records of this ancient era are difficult to track down and I will have to resort to a failing memory. Most details have become blurred and only a few highlights stand out.

It all began in the dim and distant past when I was approached by Lord Flowers, at that time Rector of Imperial College, London, and President of the European Science Foundation, who suggested to me that we mathematicians should copy the physicists in establishing our own European society. As a physicist, Flowers was no doubt involved in the establishment and operation of the EPS, and knew that such a body was needed to liaise with the various organisations that were growing up in Europe, notably the fledgling Parliament and Economic Union.

I agreed to take on the task, little thinking that it would turn into a marathon project lasting over a decade. I do not remember the exact process but I imagine I first took the matter to the London Mathematical Society and that other societies in Europe were then approached. We met and agreed to work toward setting up an EMS, operating informally until we could establish a constitution. Naively, we expected that this process would take only a short time but, alas, this was to underestimate mathematicians. In principle, one would think that mathematicians, with their clear, logical minds, would have quickly solved the simple problem of devising a constitution. However, mathematicians like to dig into foundations and be clear on principles. It gradually became clear that there were widely different views on the nature and function of a future EMS and what its relation would be to already existing national societies.

Meeting in Liblice, Poland, November 1986. Sir Michael Atiyah is 5th from the left.

Essentially, there were two, diametrically opposite views and the split was, by and large, along national lines, with France and Germany in particular holding different views. The French wanted a society consisting of individual members, while the Germans wanted a federation of national societies. I forget now all the details of the discussions but I remember that our first attempt to produce a draft constitution failed to pass the critical meeting. However, we had already been quite active in fostering European cooperation, without having any formal status. For example, we had tried to establish a uniform approach to the computerisation of mathematics (and we even got a grant for the purpose). This was a worthy ambition and well ahead of its time, even though we eventually failed and were overtaken by events.

Faced with our failure to agree, we decided there were enough things we could usefully do to justify our continued existence as an informal body. We now called ourselves the European Mathematical Council and I was appointed chairman. We had annual meetings and, in addition to our practical activities, we still pursued the ambition to aim for an EMS.

Finally, after more than ten years, compromises were made by both sides and, at a meeting in Poland, we established the EMS. The solution was to have both individual and institutional bodies as members. I remember reflecting on the difficulties of international negotiation and the comparison between diplomats and mathematicians. International agreements are notoriously difficult, as we have seen with issues such as climate change and free trade. But politicians are experts in the art of pragmatic compromises and real world problems have to be solved somehow. Deals are struck and the world moves on. Mathematics is different; we are even better than lawyers at analysing fine detail and spotting tricky points but we dislike messy solutions. Moreover, the world does not collapse if we do not get agreement. This is fundamentally why we took so long to establish our EMS!

Once the constitution had been agreed, the EMS was born and things moved on. I decided to hand over to someone else and I was fortunate enough to be able to persuade my good friend Fritz Hirzebruch to become the first president. He was the ideal choice, not only because of his standing and organisational skills but because the

Madralin, Poland, 27–28 October 1990

reunification of Germany and the collapse of the Berlin Wall opened up the East. Throughout the existence of the EMS, we had interpreted Europe in the broad sense and I remember Georgia explaining carefully to us that, because of some famous river boundary, Georgia was in Europe and not in Asia. But it was going to be much easier and more natural for the EMS to cover the whole of the new Europe and Germany was centrally placed to assist the process.

Over the past 25 years, it has been a pleasure to watch the development of the EMS under its various presidents. I always hoped that we would succeed and that it had an important role to play. In some large countries with long mathematical traditions the national societies are strong and active but in smaller and newer countries this may not be so true and I felt that it was precisely in such smaller countries that mathematicians would benefit from being part of a European-wide organisation. I think the success of the EMS vindicates this view.

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Sir Michael Francis Atiyah is a geometer with an interest in theoretical physics. He has spent most of his academic life in the United Kingdom at Oxford and Cambridge, and in the United States at the Institute for Advanced Study. He was awarded the Fields Medal in 1966 and the Abel Prize in 2004.

Foundation of the European Mathematical Society

On Sunday 28th October 1990 at Madralin, some 20 kilometres from Warsaw and under the hospitality of the Polish Academy of Sciences, there came into existence a new Society, the European Mathematical Society. This Society has been founded under an initiative of some 30 mathematical societies drawn from virtually every country of the European continent from the Atlantic Ocean to the Ural Mountains. The Society has been founded at a historic juncture in European affairs and, for legal purposes, has been established under Finnish law with its seat in Helsinki.

The aims of the Society, as given by Article 2 of its Statutes, include:

"The purpose of the Society, is to promote the development of all aspects of mathematics in the countries of Europe, with particular emphasis on those which are best handled on an international level.

The Society will concentrate on those activities which transcend national frontiers and it will in no way seek to interfere with the national activities of the member societies.

In particular, the Society will, in the European context, aim to promote mathematical research (pure and applied), assist and advise on problems of mathematical education, concern itself with the broader relations of mathematics to society, foster the interaction between mathematicians of different countries, establish a sense of identity amongst European mathematicians, and represent the mathematical community in supra-national institutions."

Among the immediate active concerns being investigated is the possibility of a newsletter and of a mathematical journal.

Reflecting the manner in which the Society has been set up the membership rules are somewhat complicated. The original 30 or so founding mathematical societies are deemed to have joined as **full members** of the new Society; other societies are respectfully invited to join but **full membership** is restricted to those organisations primarily concerned with promoting research in pure or applied mathematics within Europe. A private individual, who makes a contribution to European mathematics, is cordially invited to become an **individual member**, either by joining the Society directly or by joining via membership of some society which itself is a **full member**. The significance of the difference in the mode of membership for a private individual is that by joining directly he or she will pay 280 Finnish marks annually whereas by joining via a society he or she will pay only 70 Finnish marks annually [one US dollar = 3.6 Finnish marks approximately]; members may expect to receive a 50% reduction on the registration fee

1

for a major Congress which is planned for Paris in the early Summer of 1992. Provision, with appropriate fees, exists for **associate members**, these being societies such as teacher organisations with an interest in mathematics and for **institutional members** such as commercial organisations, industrial laboratories or academic institutions.

At its first meeting the fledgling Society elected the following as office- bearers and members of its Executive Committee:

The Officers of the Society are:

President:

Professor F. Hirzebruch, Bonn, Germany

Vice-Presidents:

Professor Cz. Olech, Warsaw, Poland,

Professor A. Figà-Talamanca, Rome, Italy

Secretary:

Professor C. Lance, Leeds, United Kingdom

Treasurer:

Professor A. Lahtinen,Helsinki, Finland

Members:

Professor E. Bayer, Geneva, Switzerland

Professor A. Kufner, Prague, Czechoslovakia

Professor P.-L. Lions, Paris IX, France

Professor L. Márki, Budapest, Hungary

Professor A. St. Aubyn, Lisbon, Portugal

Long may the Society flourish!

(*Publicity: Professor D.A.R. Wallace, Strathclyde, Glasgow, United Kingdom*)

2

Official announcement of the foundation of the European Mathematical Society

The *h*-Principle and Onsager's Conjecture

Camillo De Lellis (Universität Zürich, Switzerland) and László Székelyhidi (Universität Leipzig, Germany)

1 Introduction

The *h*-principle is a concept introduced by Gromov that pertains to various problems in differential geometry where one expects high flexibility of the moduli spaces of solutions due to the high-dimensionality (or underdetermined nature) of the problem. Interestingly, in some cases a form of the *h*-principle holds even for systems of partial differential relations that are, formally, not underdetermined.

Perhaps the most famous instance is the Nash–Kuiper theorem on C^1 isometric Euclidean embeddings of n-dimensional Riemannian manifolds. In the classical situation of embedding (two-dimensional) surfaces in three-space, the resulting maps comprise three unknown functions that must satisfy a system of three independent partial differential equations. This is a determined system and, indeed, sufficiently regular solutions satisfy additional constraints (C^2, i.e. continuous second order derivatives, suffices). The oldest example of such a constraint is the Theorema Egregium of Gauss: the determinant of the differential of the Gauss map (a priori an "extrinsic" quantity) equals a function which can be computed directly from the metric, i.e. the intrinsic Gauss curvature of the original surface.

At a global level there are much more restrictive consequences: for instance any (C^2) isometric embedding u of the standard 2-sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ can be extend in a unique way to an isometry of $\mathbb{R}^3$ and must therefore map $\mathbb{S}^2$ affinely onto the boundary of a unitary ball. In other words C^2 isometric embeddings of the standard 2-sphere in $\mathbb{R}^3$ are rigid; in fact the same holds for any metric on the 2-sphere which has positive Gauss curvature.

Nevertheless, the outcome of the Nash–Kuiper theorem is that C^1 solutions are very flexible and all forms of the aforementioned rigidity are lost. In a sense, in this situation low regularity serves as a replacement for high-dimensionality.

A similar phenomenon has been found recently for solutions of a very classical system of partial differential equations in mathematical physics: the Euler equations for ideal incompressible fluids. Regular (C^1) solutions of this system are determined by boundary and initial data, whereas continuous solutions are not unique and might even violate the law of conservation of kinetic energy. Although at a rigorous mathematical level this was proved only recently, the latter phenomenon was predicted in 1949 by Lars Onsager in his famous note [41] about statistical hydrodynamics. Onsager conjectured a threshold regularity for the conservation of kinetic energy. The conjecture is still open and the threshold has deep connections with Kolmogorov's theory of fully developed turbulence.

In this brief note, we will first review the isometric embedding problem, emphasising the *h*-principle aspects. We will then turn to some "*h*-principle-type statements" in the theory of differential inclusions, proved in the last three decades by several authors. These results were developed independently of Gromov's work but a fruitful relation was pointed out in a groundbreaking paper by Müller and Šverak 15 years ago (see [38]). There is, however, a fundamental difference: in differential geometry, the *h*-principle results are in the "C^0 category", whereas the corresponding statements in the theory of differential inclusions hold in the "L^∞ category". Indeed, "L^∞ *h*-principle statements" in differential geometry are usually trivial, whereas "C^0 *h*-principle statements" in the theory of differential inclusions are usually false. Surprisingly, both aspects are present and nontrivial when dealing with solutions of the incompressible Euler equations. The last two sections of this note will be devoted to them.

2 Nash and the isometric embedding problem

Let M^n be a smooth, compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An isometric embedding of (M^n, g) into $\mathbb{R}^m$ is a continuous map u which preserves the length of curves. Obviously, this implies that u is a Lipschitz homeomorphism of M and $u(M)$. For $u \in C^1$, the length-preserving condition amounts, in local coordinates, to the system

$$\partial_i u \cdot \partial_j u = g_{ij}\,, \tag{1}$$

consisting of $n(n+1)/2$ equations in m unknowns. The system obviously guarantees that any C^1 solution is an immersion: the property of being an embedding is than simply equivalent to the injectivity of the map u. We will therefore use the term "isometric immersion" for C^1 solutions of (1) that are not necessarily injective.

We note in passing that one may also study "weak" solutions of system (1). Recall that a Lipschitz mapping $u : M \to \mathbb{R}^m$ is, in virtue of the classical Rademacher theorem, differentiable almost everywhere. Then, we say that $u \in Lip$ is a weak isometry if (1) holds almost everywhere on M. However, being a weak isometry does not imply that the length of curves is preserved. As pointed out by Gromov in [29], such a map may – and in fact, generically will (see [36]) – contract whole submanifolds of M into single points.

Before the fundamental works of Nash in the 1950s, only the existence of local analytic embeddings for analytic metrics was known (for $m = \frac{n(n+1)}{2}$) – see [32] and [11]. Assuming for the moment that $g \in C^\infty$, the pioneering ideas introduced

by Nash culminated in two "classical" theorems concerning the (global!) solvability of (1):

Theorem 1 (Nash [40], Gromov [29])**.** Let $m \geq (n+2)(n+3)/2$ and $v : M \to \mathbb{R}^m$ be a *short* embedding (resp. immersion) of M, i.e. a C^1 embedding (resp. immersion) satisfying the inequality $\partial_i v \cdot \partial_j v \leq g_{ij}$ in the sense of quadratic forms. Then v can be uniformly approximated by isometric embeddings (resp. immersions) of class C^∞.

Theorem 2 (Nash [39], Kuiper [37])**.** If $m \geq n+1$ then any short embedding (resp. immersion) can be uniformly approximated by isometric embeddings (resp. immersions) of class C^1.

Theorems 1 and 2 are not merely existence theorems; they show that there exists a huge (essentially C^0-dense) set of solutions. This type of abundance of solutions is a central aspect of Gromov's h-principle. Naively, such "flexibility" could be expected for high codimension as in Theorem 1, since then there are many more unknowns than equations in (1). The h-principle for C^1 isometric embeddings is, on the other hand, rather striking, especially when compared to the classical rigidity result concerning the Weyl problem (see [43] for a thorough discussion):

Theorem 3 (Cohn Vossen [14], Herglotz [30])**.** If $(\mathbb{S}^2, g)$ is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$ then $u(M)$ is uniquely determined up to a rigid motion.

It is intuitively clear that weak (i.e. Lipschitz) isometries cannot enjoy any rigidity property of this type. One can think, for instance, of folding a piece of paper. The folding preserves length, and is therefore isometric, but the resulting map is clearly not C^1: the tangent vector is not continuous across folds. The difficulty of the Nash–Kuiper theorem is precisely to obtain a continuous tangent vector and this requires a complicated "high dimensional" construction.

Thus, it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

Theorems 1 and 2 make use of a certain extra freedom or "extra dimensions" in the problem. The proof of Theorem 1 relies on the Nash-Moser implicit function theorem and yields solutions which are not only isometric but also *free* – the $n + n(n+1)/2$ vectors of first and second partial derivatives of the map u are linearly independent in $\mathbb{R}^m$ at each point x. The presence of "extra dimensions" in the proof of Theorem 2 is more subtle and manifests itself as low regularity. Naively, one might think of low regularity in this context as having a large number of active Fourier modes.

The iteration technique in the proof of Theorem 2, called *convex integration*, was subsequently developed by Gromov [28, 29] into a very powerful and very general tool to prove the h-principle in a wide variety of geometric-topological problems (see also [25, 45]). In such situations, the sought-after solution must typically satisfy a pointwise *inequality* rather than an equality. An example is to find n divergence-free vector fields on a parallelisable n-dimensional manifold that are linearly independent at any point – the inequality here arises from the pointwise linear independence. Convex integration in this context is essentially a homotopic-theoretic method. In contrast, for *equalities* there is no general method except in certain cases (so-called ample relations), which do not include Theorem 2 or the applications to fluid mechanics below.

In general, the regularity of solutions obtained using convex integration (for ample relations) agrees with the highest derivatives appearing in the equations (see [44]). An interesting question raised in [29, p. 219] is how one could extend convex integration to produce more regular solutions. Essentially the same question, in the case of isometric embeddings, is also mentioned as Problem 27 in [46]. In the latter context, for high codimension this was resolved by Källen in [33]. In codimension 1 the problem was first considered by Borisov, who in [6] announced that if g is analytic then the h-principle holds for local isometric embeddings $u \in C^{1,\alpha}$ with $\alpha < \frac{1}{1+n+n^2}$ ($C^{k,\alpha}$ is the usual notation for spaces of C^k maps u such that each partial derivative w of order k is Hölder continuous with exponent α, namely satisfying the bound $|w(x) - w(y)| \leq C(d(x,y))^\alpha$, where d is the Riemannian distance). A proof for the case $n = 2$ appeared in [7]; for a proof in any dimension, also valid for C^2 metrics, the reader is referred to [16]. Borisov also pointed out that the optimal regularity for rigidity statements is *not* C^2: in particular, in a series of papers [1, 2, 3, 4, 5] he showed that Theorem 3 is valid for $C^{1,\alpha}$ isometric immersions u when $\alpha > \frac{2}{3}$ (see [16] for a short proof).

3 The h-principle as a relaxation statement

The h-principle amounts to the vague statement that local constraints do not influence global behaviour. In differential geometry, this leads to the fact that certain problems can be solved by purely topological or homotopic-theoretic methods, once the "softness" of the local (differential) constraints has been shown. In turn, this softness of the local constraints can be seen as a kind of relaxation property.

In order to gain some intuition, let us again look at the system of partial differential equations (1) with some fixed, smooth g. Obviously, any sequence of solutions

$$\{u^k\}_k\,, \qquad u^k : \Omega \to \mathbb{R}^m$$

of (1) enjoys a uniform bound upon the maximum of $|\partial_i u^k|$ and thus the Arzelà-Ascoli theorem guarantees uniform convergence, up to subsequences, to some limit map u. The limit u must be Lipschitz and an interesting question is whether we can recover some better convergence from the equations, for instance in the C^1 category. As we have learned from the previous section, this depends on the codimension and the a priori assumptions on the smoothness of the sequence. For instance, for surfaces in 3-space, if the metric g has positive curvature and the maps u^k are sufficiently smooth, their images will be (portions of) convex surfaces; this, loosely speaking, amounts to some useful information about second derivatives which will improve the convergence of u^k and result in a limit u with convex image.

If instead we only assume that the sequence u^k consists of approximate solutions, for instance in the sense that

$$\partial_i u^k \cdot \partial_i u^k - g_{ij} \to 0 \text{ uniformly},$$

then even if g has positive curvature and the u^k are smooth, their images will not necessarily be convex. Let us, nonetheless, see what we can infer about the limit u. Consider a smooth curve $\gamma \subset \Omega$. Then $u^k \circ \gamma$ is a C^1 Euclidean curve. As already noticed, if we denote by $L_e(u^k \circ \gamma)$ the "Euclidean length" and by $L_g(\gamma)$ the length of γ in the Riemannian manifold (Ω, g) then

$$L_e(u^k \circ \gamma) - L_g(\gamma) \to 0. \tag{2}$$

On the other hand, the curves $u^k \circ \gamma$ converge uniformly to the (Lipschitz) curve $u \circ \gamma$ and it is well-known that under such types of convergence the length might shrink but cannot increase. In other words, we conclude that

$$L_e(u \circ \gamma) \leq L_g(\gamma)\,. \tag{3}$$

Recall that, by Rademacher's theorem, u is differentiable almost everywhere: it is a simple exercise to see that when (2) holds for every curve γ in Ω, then

$$\partial_i u \cdot \partial_j u \leq g_{ij} \qquad \text{a.e. in } \Omega \tag{4}$$

(as above, the latter inequality should be understood in the sense of quadratic forms).

Thus, loosely speaking, one possible interpretation of Theorems 1 and 2 is that the system of partial differential inequalities (4) is the "relaxation" of (1) (resp. in the C^∞ and C^1 categories) with respect to the C^0 topology. In order to explain this better, let us simplify the situation further and consider the case $\Omega \subset \mathbb{R}^n$ with the flat metric $g_{ij} = \delta_{ij}$, to be embedded isometrically into $\mathbb{R}^m$. Then, the system (1) is equivalent to the condition that the full matrix derivative $Du(x)$ is a linear isometry at every point x, i.e. that

$$Du(x) \in O(n, m) \tag{5}$$

for every x. Note also that the inequality (4) is similarly equivalent to

$$Du(x) \in \text{co}\, O(n, m), \tag{6}$$

where for a compact set K we denote its convex hull by co K.

More generally, given a compact set of matrices $K \subset \mathbb{R}^{m\times n}$, one considers the differential inclusion

$$Du(x) \in K \tag{7}$$

and its relaxation – the latter may be given by the convex hull co K but it might also be a strictly smaller set. The local aspect of the h-principle amounts to the statement that solutions of the original inclusion (7) are dense in C^0 in the potentially much larger relaxation.

This might seem very surprising but consider the following one-dimensional problem, i.e. the case $n = m = 1$. Thus, setting $\Omega = [0, 1]$, we are looking at the inclusion problem $u'(x) \in \{-1, 1\}$. Of course C^1 solutions need to have constant derivative ± 1 but Lipschitz solutions may be rather wild. In fact, it is not difficult to show that the closure in C^0 of the set

$$\mathcal{S} := \{u \in \text{Lip}[0, 1] : \ |u'| = 1 \text{ a.e.}\}$$

coincides with the convex hull

$$\mathcal{R} := \{u \in \text{Lip}[0, 1] : \ |u'| \leq 1 \text{ a.e.}\}\,.$$

Since the topology of uniform convergence in this setting (uniform Lipschitz bound) is equivalent to weak* convergence of the derivative in L^∞, the latter statement can be interpreted as a form of the Krein-Milman theorem. Moreover, it was observed in [12] that $\mathcal{R} \setminus \mathcal{S}$ is a meagre set in the Baire Category sense (see also [8]).

For general differential inclusions with $m, n \geq 2$, the situation is more complicated but there is – as a rule of thumb – a kind of dichotomy, depending on the set K. Either

(a) one has a large relaxation and a Krein-Milman type result as above; or

(b) one has rigidity (and essentially "no relaxation").

These two situations have been studied in detail in the context of nonlinear elasticity (see [17, 38, 34, 35]). Case (a) can be interpreted as a form of the h-principle, albeit a weak form as, in general, the solutions to the corresponding problem (7) will be Lipschitz but not necessarily C^1.

As discussed above, in the weak isometric map problem (i.e. the case where $K = O(n, m)$), solutions can be intuitively constructed by folding (see also [18, 19]): such maps have an altogether different structure from the Nash–Kuiper C^1 solutions. In this example, the existence of many Lipschitz solutions is not as surprising as Theorem 2. Next, we discuss the Euler equations, where a weak form of the h-principle is already rather striking.

4 The Euler equations as a differential inclusion

The incompressible Euler equations are perhaps the oldest system of partial differential equations in fluid dynamics, derived by Euler more than 250 years ago. The unknowns are the velocity v of the fluid and the (mechanical) pressure field p. Both of them depend upon a space variable x (ranging in some domain Ω of $\mathbb{R}^2$ and $\mathbb{R}^3$ or in the periodic tori $\mathbb{T}^2$, $\mathbb{T}^3$) and a time variable t. The system can be written as:

$$\begin{cases} \partial_t v + \text{div}_x(v \otimes v) + \nabla p = 0, \\ \text{div}_x\, v = 0\,. \end{cases} \tag{8}$$

For C^1 functions v, the nonlinearity $\text{div}_x(v \otimes v)$ equals the advective derivative $(v \cdot \nabla)v$, which in components is expressed as

$$[(v \cdot \nabla)\, v]_i = \sum_j v_j \partial_j v_i\,.$$

The reader familiar with the theory of distributions will recognise that, after writing the Euler equations as in (8), we can naturally introduce a concept of weak solutions as soon as v is a square summable function. We refrain, however, from defining such "distributional solutions" formally. Rather, we describe a possible route to such a concept.

Assume for the moment that the pair (v, p) is smooth and satisfies (8). Consider a "fluid element", namely a region $U \subset\subset \Omega$ with smooth boundary ∂U. If we integrate the second equation of (8) in the space variable and use the divergence theorem, we achieve

$$\int_{\partial U} v(x, t) \cdot n(x)\, dS\,(x) = 0\,, \tag{9}$$

where n denotes the outward unit normal to ∂U (and we use the notation $\int_\Sigma f(x)\, dS\,(x)$ for surface integrals). If we instead

integrate the first equation on $U \times [a,b]$, we then achieve

$$\int_a^b \left[\int_{\partial U} \left((v(x,t) \cdot n(x)) v(x,t) + p(x,t) n(x) \right) dS(x) \right] dt = \int_U (v(x,a) - v(x,b))\, dx. \quad (10)$$

Both identities make perfect sense when (v,p) are merely continuous functions and express the balance of mass and momentum for the portion of the fluid which occupies the region U. Equation (9) simply expresses the conservation of mass, since it requires that the number of fluid particles leaving U balances that of particles entering U. Equation (10) expresses the variation of the momentum, which can change for only two reasons:

– Particle fluids leave the region U, carrying different momenta compared to those entering.
– The fluid occupying the "external regions", namely the complement of the fluid element U, exerts a force on the portion occupying U; such force is directed along the unit normal to the boundary ∂U as it is proportional to the mechanical pressure p.

It is not difficult to see that *continuous* (v,p) are distributional solutions of (8) if and only if the identities (9) and (10) hold for every smooth $U \subset\subset \Omega$. However, the weak formulation through (9)-(10) is very natural and interesting per se: it is indeed common to derive the equations governing a continuous system by first considering the laws of conservation of motion in different regions. The corresponding partial differential equations are then derived following a process which is the reverse of the one outlined above.

Finally, if we wish to abandon the requirement that (v,p) is continuous, general distributional solutions can be suitably characterised as maps satisfying (9) and (10) for "almost all" fluid elements U.

The system (8) is, for classical C^1 solutions, deterministic: when supplied with appropriate boundary conditions such solutions are unique. The most common condition (when the space domain $\Omega = \mathbb{R}^2, \mathbb{R}^3, \mathbb{T}^2, \mathbb{T}^3$ and the space-time domain is $\Omega \times [0,T]$) is the initial value

$$v(\cdot, 0) = v_0\,.$$

Classical solutions are then uniquely determined by the initial data v_0 (but in the cases $\Omega = \mathbb{R}^2, \mathbb{R}^3$ some additional assumption upon the decay at spatial infinity is needed: $v(\cdot,t) \in L^2$ is the most natural one and it is sufficient).

Surprisingly, Scheffer proved in [42] that the situation is completely different for irregular weak solutions.

Theorem 4 (Scheffer 1993)**.** There is a nontrivial, compactly supported $v \in L^2(\mathbb{R}^2 \times \mathbb{R})$ which solves (8) in the sense of distributions.

In [22], we have shown that the latter theorem can be derived very naturally as a corollary of a suitable h-principle statement, or relaxation result, in the spirit of the previous section. There are several powerful general versions of such statements, which severely restrict natural attempts to give a definition of "admissible weak solutions" enjoying uniqueness (see for instance [23]). To keep our discussion as simple as possible, here we restrict to a rather easy version. But we first need to introduce the system of "partial differential inequalities" which is the appropriate relaxation of (8).

Definition 5 (Subsolutions)**.** Let $\overline{e} \in C^\infty \cap L^1(\mathbb{R}^n \times \mathbb{R})$ with $\overline{e} \geq 0$. A triple of smooth, compactly supported functions

$$(v,u,q) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}$$

is a subsolution of (8) with energy density $\bar{e}$ if the following properties hold:

(i) u takes values in the subspace of symmetric trace-free matrices and $\operatorname{spt}(v,u,q) \subset \operatorname{spt}(\bar{e})$.

(ii) (v,u,q) solves

$$\begin{cases} \partial_t v + \operatorname{div} u + \nabla q = 0, \\ \operatorname{div} v = 0. \end{cases} \quad (11)$$

(iii) The following inequality holds, in the sense of quadratic forms, on the set $\{\bar{e} > 0\}$

$$v_i v_j - u_{ij} < \tfrac{2}{n} \overline{e} \delta_{ij}\,. \quad (12)$$

Theorem 6 ([22])**.** Let $\overline{e} \in C^\infty \cap L^1(\mathbb{R}^n \times \mathbb{R})$ and $(\overline{v}, \overline{u}, \overline{q})$ be a subsolution with kinetic energy $\overline{e}$. Then there exists a sequence of bounded weak solutions (v^k, p^k) of (8) on $\mathbb{R}^n \times \mathbb{R}$ such that

$$\tfrac{1}{2}|v^k|^2 = \overline{e} \quad \text{a.e.} \quad (13)$$

and $v^k \to v$ weakly in L^2.

The analogy with the aforementioned results in the theory of differential inclusions is rather striking. But, perhaps more surprisingly, in the case of the Euler equations a similar statement can be proved in the C^0 category.

Theorem 7 ([24])**.** Let $E \in C^\infty([0,T])$ with $E > 0$. Then there exists a sequence of continuous weak solutions (v^k, p^k) of (8) on $\mathbb{T}^3 \times [0,T]$ such that

$$\tfrac{1}{2} \int |v^k|^2(x,t)\, dx = \overline{E}(t) \quad \text{for every } t \quad (14)$$

and $v^k \to 0$ weakly in L^2.

In fact, it is possible to extend the preceeding theorem and produce sequences which converge not to zero but to a vector-field v from a certain class of "subsolutions", as in Theorem 6. However, presently there is no full characterization of the corresponding "relaxed problem" (for some results in this direction see however [20]) . Note also that Theorem 7 remains valid in two space dimensions as well (see [13]).

From Theorem 7, we conclude that continuous solutions of the Euler equations do not necessarily preserve kinetic energy. This phenomenon was, in fact, predicted long ago by Lars Onsager and we will discuss it in the next section.

5 Onsager's conjecture and continuous dissipative solutions

One of the fundamental problems in the theory of turbulence is to find a satisfactory mathematical framework linking the basic continuum equations of fluid motion to the highly chaotic, apparently random behaviour of fully developed turbulent flows. Consider the incompressible Navier–Stokes equations

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \mu \Delta v\,, \\ \operatorname{div} v = 0\,, \end{cases} \quad (15)$$

describing the motion of an incompressible viscous fluid. The coefficient $\mu > 0$ is the viscosity, which, after appropriate non-dimensionalising, equals the reciprocal of the Reynolds number Re. As μ becomes smaller (or, more precisely, the Reynolds number becomes larger), the observed motion becomes more and more complex, at some stage becoming chaotic. The statistical theory of turbulence, whose foundations were laid by Kolmogorov in 1941, aims to describe universal patterns in this chaotic, turbulent flow sufficiently far away from the domain boundaries by postulating that generic flows can be seen as realisations of random fields and by using the symmetry and scaling properties of the Navier–Stokes equations; we refer the reader to [27].

One of the cornerstones of the theory is the famous Kolmogorov-Obukhov 5/3 law. It states that the energy spectrum $E(k)$, defined to be the kinetic energy per unit mass and unit wavenumber, behaves like a power law

$$E(k) \sim k^{-5/3}. \tag{16}$$

This power law, which is supposed to be valid in a certain intermediate range of wave numbers k – called the inertial range – away from the large scales (affected by the boundaries of the domain and external forces) and away from the very small scales (affected by dissipation), agrees remarkably well with experiments and numerical simulations. Closely related to the 5/3 law is the idea of an energy cascade, originally due to Richardson. The energy is introduced at large scales and, through nonlinear interaction, it cascades to smaller and smaller scales until it is dissipated by the viscosity in the very small scales (see [27]). Indeed, a key hypothesis of the K41 theory is that the mean rate of energy dissipation ϵ is strictly positive and independent of μ in the infinite Reynolds number limit ($\mu \to 0$). This effect in turbulent flows is known as *anomalous dissipation*.

Extending the inertial range to infinitely small scales (i.e. $k \to \infty$) corresponds in a certain sense to the limit $\mu \to 0$, when (15) becomes the incompressible Euler equations

$$\begin{cases} \partial_t v + \operatorname{div}_x(v \otimes v) + \nabla p = 0, \\ \operatorname{div}_x v = 0\,. \end{cases} \tag{17}$$

A classical calculation shows that for smooth solutions (v, p) of (17) the kinetic energy is conserved

$$\int |v(x,t)|^2\, dx = \int |v(x,0)|^2\, dx. \tag{18}$$

Lars Onsager suggested in his famous note [41] the possibility of anomalous dissipation for *weak solutions* of the Euler equations as a consequence of the energy cascade. It is worth emphasising that, although the K41 theory and the theory of turbulence in general is a *statistical theory* concerned with ensemble averages of solutions of the Navier–Stokes equations, the suggestion of Onsager turns this into the following "pure PDE" question:

Conjecture 1. For weak solutions (v, p) of (17) with

$$|v(x,t) - v(y,t)| \le C|x-y|^\theta \quad \forall x, y, t \tag{19}$$

(where the constant C is independent of x, y, t) we have:

(a) For $\theta > 1/3$, the energy is conserved by any solution, i.e. (18) holds.

(b) For $\theta < 1/3$, there are solutions which do not conserve the energy.

The space of functions satisfying (19) is usually denoted $L^\infty(0,T;C^\theta(\mathbb{T}^3))$ and belongs naturally to the hierarchy of spaces $L^p(0,T;C^\theta(\mathbb{T}^3))$: a function in the latter space is assumed to satisfy (19) at a.e. t with a time-dependent constant $C(t)$ such that $\int C(t)^p dt < \infty$.

The first part of the conjecture, i.e. assertion (a), has been shown by Eyink in [26], following some original computations of Onsager, and by Constantin, E and Titi in [15]. The proof amounts to giving a rigorous justification of the formal computation leading to (18) and, in [15], this is done via a suitable regularisation of the equation and a commutator estimate (whereas Onsager's original calculations are based on convergence of Fourier series).

Concerning the second part of the conjecture, clearly the first mathematical statement in that direction is Theorem 4. Theorem 7 showed for the first time rigorously that C^0 solutions can *dissipate* the kinetic energy. The two statements are prototypical of a series of recent results concerning point (b) of Conjecture 1, which take the techniques of [24] as a starting point. Therefore, having fixed a certain specific space of functions X, these results can be classified in the following two categories:

(A) There exists a nontrivial weak solution $v \in X$ of (17) with compact support in time.

(B) Given any smooth positive function $E = E(t) > 0$, there exists a weak solution $v \in X$ of (17) with

$$\int_{\mathbb{T}^3} |v(x,t)|^2\, dx = E(t) \quad \forall\, t.$$

Obviously, both types lead to non-conservation of energy and would therefore conclude part (b) of Onsager's conjecture if proved for the space $X = L^\infty(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$ for every $\epsilon > 0$. So far, the best results are as follows.

Theorem 8. Let ϵ be any positive number smaller than $\frac{1}{5}$. Then

– Statement (A) is true for $X = L^1(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$.

– Statement (B) is true for $X = L^\infty(0,T;C^{1/5-\epsilon}(\mathbb{T}^3))$.

Statement (B) has been shown for $X = L^\infty(0,T;C^{1/10-\epsilon})$ in [21], whereas P. Isett in [31] was the first to prove Statement (A) for $X = L^\infty(0,T;C^{1/5-\epsilon})$, thereby reaching the current best "uniform" Hölder exponent for Part (b) of Onsager's conjecture. Subsequently, T. Buckmaster, the two authors and P. Isett have proved Statement (B) for $X = L^\infty(0,T;C^{1/5-\epsilon})$ in [9]. Finally, Statement (A) for $X = L^1(0,T;C^{1/3-\epsilon}(\mathbb{T}^3))$ has been proved very recently in [10].

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Camillo De Lellis [camillo.delellis@math.uzh.ch] is a full professor at the University of Zürich. He studied mathematics at the University of Pisa and the Scuola Normale Superiore and his main research interests are in geometric measure theory and the calculus of variations, with ramifications on PDEs and geometric analysis.

László Székelyhidi Jr. [szekelyhidi@math.uni-leipzig.de] is a full professor at the University of Leipzig. He studied mathematics at the University of Oxford and at the Max-Planck Institute for Mathematics in the Sciences in Leipzig and his main research interests are in the calculus of variations, partial differential equations and hydrodynamics.

Cannons at Sparrows

Günter M. Ziegler (Freie Universität Berlin, Germany)

The story told here starts with an innocuous little geometry problem, posed in a September 2006 blog entry by R. Nandakumar, an engineer from Calcutta, India. This little problem is a "sparrow": it is tantalising, it is not as easy as one could perhaps expect and it is recreational mathematics – of no practical use.

I will sketch, however, how this little problem connects to very serious mathematics. For the modelling of this problem, we employ insights from a key area of applied mathematics, the theory of optimal transportation. This will set the stage for application of a major tool from very pure mathematics, known as equivariant obstruction theory. This is a "cannon" and we'll have some fun firing it at the sparrow.

On the way to the solution, combinatorial properties of a very classical geometric object, the permutahedron, turn out to be essential. These will, at the end of the story, lead us back to India, with some time travel that takes us 100 years into the past. For the last step in our (partial) solution of the sparrow problem, we need a simple property of the numbers in Pascal's triangle, which was first observed by Balak Ram in Madras, 1909.

But even if the existence problem is solved, the little geometry problem is not. If the solution exists, how do you find one? This problem will be left to you. Instead, I will comment on the strained relationship between cannons and sparrows and avail myself of a poem by Hans Magnus Enzensberger.

A sparrow

On Thursday 28 September 2006, very early in the morning at 6:57 am, the engineer R. Nandakumar, who describes himself as "Computer Programmer, Student of Mathematics, Writer of sorts", posts on his blog "Tech Musings" (nandacumar.blogspot.de) the following plane geometry conjecture, which he designed together with his friend R. Ramana Rao:

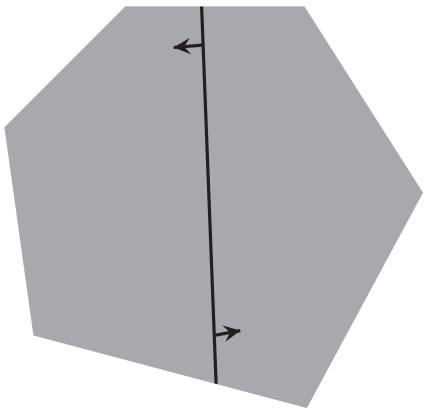
> Given any convex shape and any positive integer N. There exist some way(s) of partitioning this shape into N convex pieces so that all pieces have equal area and equal perimeter.

This problem looks entirely harmless. Perhaps it could be a high school geometry problem? Perhaps it sounds like a Mathematical Olympiad problem? Think about it yourself! For this, you might take the shape to be a triangle and let $n = 3$ or $n = 6$. Or, as Nandakumar notes, it is not clear how to divide even an equilateral triangle into $n = 5$ convex pieces of equal area and perimeter. Any ideas? Try!

The problem caught the attention of the computational geometry community after Nandakumar posted it on "Open Problem Garden" (openproblemgarden.org) in December 2007. Then, on 11 December 2008, Nandakumar and Ramana Rao announced their first piece of progress on arXiv (arxiv:0812.2241):

> We show a simple proof that the answer to this question is "Yes" for n=2 (2 pieces) and give some arguments which strongly indicate that the answer is again "Yes" for n=3.

Let's do it for $n = 2$. Every division of a convex polygon into two convex pieces arises from one straight cut. We observe that there is a (unique!) vertical line that divides our polygon into two convex pieces of equal area. In general we will not be lucky, so the perimeter of the shape to its right will be larger, say, than the perimeter of the part to the left.



Now we rotate the bisecting line. Check (!) that if we rotate in such a way that we always bisect the area then the perimeters of the parts to the right and to the left of the line will change *continuously*. Thus also the quantity "perimeter to the right minus perimeter to the left" changes continuously. After we have rotated by a half-turn of 180 degrees, the value of the quantity has turned to its negative. If it was initially positive then it is negative after the half-turn and, by continuity, we must have "hit zero" somewhere in between. So a "fair" partition into $n = 2$ pieces exists according to the *intermediate value theorem*.

So the problem is solved, but how have we solved it? We have figured out that the *configuration space* of all divisions into two convex pieces is a circle (parameterised by the angle of the dividing line). And we have used continuity and applied a topological theorem, the *intermediate value theorem*. From the viewpoint of topology, this is the $d = 1$ case of the fact that there is no continuous map between spheres $S^d \to S^{d-1}$ that maps opposite points on S^d to opposite points on S^{d-1}, the *Borsuk–Ulam theorem* [13].

The problem looked harmless but topology has crept in, even for the easy case $n = 2$. Nandakumar and Ramana Rao did not have a proof for $n = 3$. However, a few days after their preprint, on 16 December 2008, Imre Bárány, Pavle Blagojević and András Szűcs submitted a solution for $n = 3$ to *Advances in Mathematics*. It was published online in September 2009 and printed in 2010 as a 15 page *Advances* paper [2]. This may already prove that the harmless little "sparrow" problem is harder than it looked at first sight.

The cannons, I

What does the space of all partitions of a polygon into 3 convex pieces look like? Blagojević et al. [2] described it as a part of a Stiefel manifold. How about the space of equal-area partitions into n pieces? We don't really understand it at all! (León [12] will offer some insights.) Here, an Ansatz from the theory of *Optimal Transport* comes to the rescue – this was first observed by Roman Karasev from Moscow.

Optimal transport is an old subject, started by the French engineer Gaspard Monge in 1781. The key result we need was established by Leonid Kantorovich in the late 1930s. (For his work Kantorovich got a Stalin Prize in 1949 and an Economics Nobel Prize in 1975.) The area is very much alive, as witnessed by two recent, major books by Cédric Villani (Fields Medallist 2010).

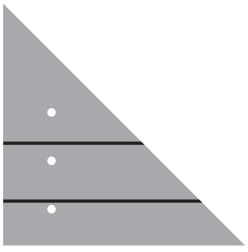
Our problem is solved by the construction of "weighted Voronoi diagrams": given any mass (the area of a convex polygon) and a set of sites with weights (that is, n distinct points and real numbers attached to them which should sum to zero), we associate to each site all the points in the polygon for which "distance to the site squared minus the weight of the site" is minimal. This yields a partition of the convex polygon into convex pieces!

Here is the result we need:

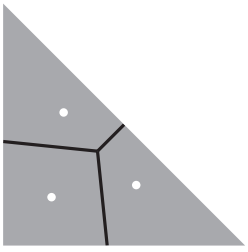
Theorem (Kantorovich [1938], etc.). *For any $n \geq 2$ distinct points in the plane, and an arbitrary polygon, there are* unique *weights such that the weighted Voronoi diagram for these points and weights subdivides the polygon into n convex pieces of the same area.*

In other words, the configuration space $F(\mathbb{R}^2, n)$ of all n-tuples of distinct points in the plane parameterises weighted Voronoi partitions into n convex pieces of equal area. Why is this helpful? Because we understand the space $F(\mathbb{R}^2, n)$ very well!

Let the polygon, for example, be a triangle and let $n = 3$. If the three points lie on a vertical line then the equal-area partition will look like this:



If we are, however, lucky to choose the right three points then we might end up with an equal-area and equal-perimeter partition:



But is there always a "lucky choice" for the n points?

A comment

Optimal transport is "very practical stuff", which contributes a lot to operations research. This is exemplified "graphically", for example, by the work of John Gunnar Carlsson, Department of Industrial and Systems Engineering, University of Southern California:[1]

or of Peter Gritzmann, Department of Mathematics, TU Munich, who uses it to do farmland reallocation in rural areas in Bavaria:

Mathematical Intelligencer, 2014(2). With kind permission of Springer Science+Business Media.

On the other hand, optimal transport is important for physics (and very hard mathematics, if Villani does it). So where should we place it?

1 The picture shows a map of a city, where each piece contains the same total road length in it. This is useful for companies using snowploughs or street sweepers or delivering mail who have to traverse every street in a region in an efficient way; by designing those districts, the vehicles will all have the same amount of work.

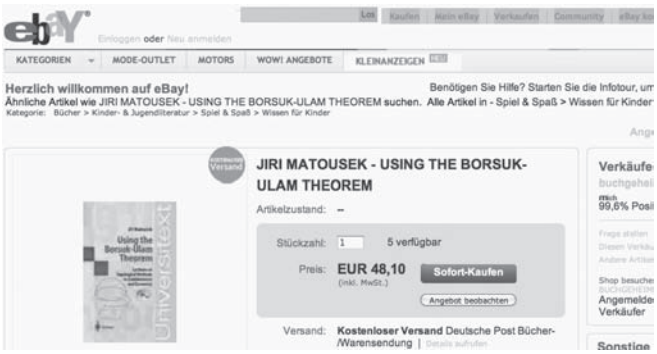
The answer is that the traditional categories plainly don't work anymore and we should discard them. What has been shown up to now is a problem from "recreational mathematics", which, for its modelling and solution, will need methods from "applied mathematics" (like optimal transport) and from "pure mathematics" (algebraic topology). Pure and applied and recreational mathematics cannot and should not be separated. There are also other parts of science that belong to mathematics *without borders*, such as (theoretical) computer science and (mathematical) operations research. If it is good science, simply call it "mathematics".

In Berlin, in the context of the Research Center MATHEON "Mathematics for Key Technologies", we try to avoid all these categories – in the end the only distinction we might make is the one between "mathematics" and "applications of mathematics".

The point of this paper is, however, a different one: within mathematics, there are "big theories" and there are "small problems". Sometimes we might need big theories to solve (apparently) small problems and we are in the course of discussing an example of this. However, at the same time, it also works in the other direction: we are using small problems to test the big theories, to see what they can do on a concrete problem. There are big theories on the shelves of university libraries for which there has never been a single concrete computation or worked out example ...

The cannons, II

The second type of cannon that we employ comes from algebraic topology. Namely, there is a well-established modelling procedure, known as the "Configuration Space/Test Map Scheme" (CS/TM), developed by Sarkaria, Živaljević and others, which converts discrete geometry problems into questions in equivariant algebraic topology. In brief, one shows that if the problem has a counterexample then there are topological spaces X and Y, where X is a *configuration space* for the problem and Y is a space of values (often a sphere), and a finite group G of symmetries such that there is a continuous map $X \to_G Y$ that preserves the symmetries. So, if an equivariant map $X \to_G Y$ does not exist then there are no counterexamples, so the problem is solved. The Borsuk–Ulam theorem, saying that there is no map $S^d \to_{\mathbb{Z}/2} S^{d-1}$, is the first major example of such a theorem. All this is beautifully explained in Jiří Matoušek's book "Using the Borsuk–Ulam Theorem" [13]. Indeed, everyone should know this, as this is "kid's stuff" – as you can see if you look for the book on *ebay*, where I found it listed under "fun and games for children"!



Well, if this kid's stuff isn't good enough for us, we will use more serious tools to treat the little polygon partition problem, namely equivariant obstruction theory. This is a method of systematically deciding whether equivariant maps $X \to_G Y$ exist. It can be seen in a wonderfully clear and precise way (without pictures or examples, though) in Section II.3 of Tammo tom Dieck's book *Transformation Groups* [8]. In order to see that there is a lucky choice for the point configurations that guarantees equal-area and equal-area partitions of our polygon – for some n – we have to interpret and evaluate the terms in the following result for our problem:

Obstruction theory 115

(3.15)). If X_1 is path-connected, then so is X_k for $k \geqq 1$. Then the first main result of obstruction theory is

(3.10) Theorem. *For each integer $n \geq 1$ there exists an exact obstruction sequence*

$$[X_{n+1}, Y]_G \to \mathrm{Im}([X_n, Y]_G \to [X_{n-1}, Y]_G) \xrightarrow{c^{n+1}} \mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$$

which is natural in (X, A) and Y.

The exactness of this sequence means that each homotopy class $X_{n-1} \to Y$ which is extendable over X_n has an associated obstruction element in the cohomology group $\mathfrak{H}_G^{n+1}(X, A; \pi_n Y)$ (as defined in (3.3)); this obstruction element is zero if and only if the homotopy class $X_{n-1} \to Y$ is extendable over X_{n+1}.

Some details

The CS/TM Scheme for our problem quite naturally leads to the following setup. If for some $n \geq 2$ and for some polygon P there is a counterexample then we get a sequence of equivariant maps:

$$\mathcal{F}(2,n) \underset{\mathfrak{S}_n}{\to} F(\mathbb{R}^2, n) \underset{\mathfrak{S}_n}{\to} \mathrm{EAP}(P, n) \underset{\mathfrak{S}_n}{\to} S^{n-2}.$$

This is a chain of very concrete objects (which are topological spaces) and unknown equivariant maps (continuous maps respecting the symmetry with respect to the group $\mathfrak{S}_n$ of permutations):

- $\mathrm{EAP}(P, n)$ is the configuration space of all partitions of P into n convex parts of equal area – a configuration space *not* well understood (see León [12]).
- S^{n-2} denotes a particular $(n-2)$-sphere, given as
$$\{y \in \mathbb{R}^n : \quad y_1 + \cdots + y_n = 0, \quad y_1^2 + \cdots + y_n^2 = 1\}$$
On this sphere, the group $\mathfrak{S}_n$ acts by simply permuting the coordinates.
- $\mathrm{EAP}(P, n) \to S^{n-2}$ maps any equal-area n-partition $(P_1, \ldots, P_n)$ to "perimeters minus average, normalised" to give a point in the sphere – under the assumption that there is no partition for which all perimeters are equal. This map is $\mathfrak{S}_n$-equivariant: permutation of the parts P_i corresponds to a permutation of the normalised perimeters and thus of the coordinates of the sphere.
- $F(\mathbb{R}^2, n)$ is the configuration space of n labelled, distinct points in the plane,
$$\{(x_1, \ldots, x_n) \in \mathbb{R}^{2\times n} : x_i \neq x_j \text{ for } i < j\}.$$
In contrast to $\mathrm{EAP}(P, n)$, this space is *pretty* well understood. It is the complement of a complex hyperplane ar-

rangement, which explains a lot of its geometry and topology; the pertinent literature starts with a classic paper from 1962 by Fox & Neuwirth [9].

- $F(\mathbb{R}^2, n) \to \mathrm{EAP}(P, n)$ is the optimal transport map, which maps $(x_1, \ldots, x_n)$ to its equal-area weighted Voronoi diagram. This is a well-defined continuous equivariant map, whose existence is due to Kantorovich (1938); an interesting recent source is Geiß, Klein, Penninger & Rote [10].
- $\mathcal{F}(2, n)$ is a finite regular cell complex model of $F(\mathbb{R}^2, n)$ – indeed, an equivariant deformation retract. Its dimension is $n-1$, it has $n!$ vertices, indexed by permutations (which correspond to point configurations in the plane where the points are sorted left-to-right according to a certain permutation) and it has $n!$ maximal cells, also indexed by permutations (which correspond to point configurations in the plane where the points lie on a vertical line, sorted according to a certain permutation). And these maximal cells have the combinatorial structure of *very* classical convex polytopes: permutahedra!
  This cell complex model was apparently first described explicitly in [7], although it can again be traced back to Fox & Neuwirth [9].
- $\mathcal{F}(2, n) \to F(\mathbb{R}^2, n)$ is again an explicit map, an equivariant inclusion.

These are all the (many) pieces in the game. The upshot is that if we assume that for some n and for some P there is no equal-area, equal-perimeter n-partition then this shows that there is an equivariant map:

$$\mathcal{F}(2, n) \underset{\mathfrak{S}_n}{\longrightarrow} S^{n-2}.$$

Does such a map exist? That's the type of question that one can answer with equivariant obstruction theory (EOS).

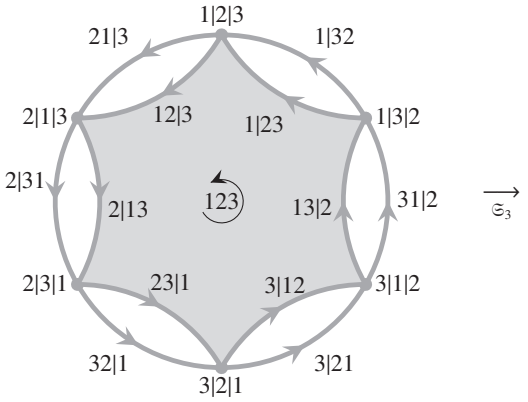
What does EOS do? It constructs the map working its way up on the dimension of the skeleton of the $(n-1)$-dimensional cell complex $\mathcal{F}(2, n)$. As we are mapping a space with a free group action into an $(n-2)$-sphere, there is no problem at all, except possibly in the last step where the map is already fixed for the boundaries ∂c_σ of the $(n-1)$-cells c_σ, which are homeomorphic to $(n-2)$-spheres.

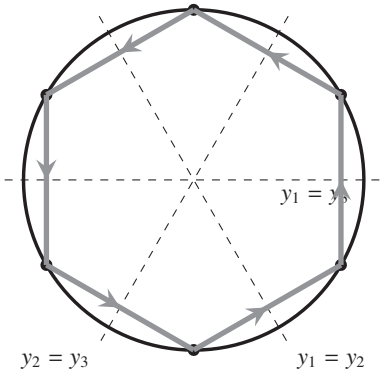
The extension is possible without any problem if all of the maps $f : \partial c_\sigma \to S^{n-2}$ have degree 0. And, indeed, all of the maps have the same degree, since we are looking for equivariant maps. However, in general, that degree won't be 0 since we may have made mistakes on lower-dimensional cells on the way to the top. EOT now says that the map can be *modified on the* $(n-2)$*-skeleton* such that the extension to the full complex is possible *if and only if* a certain equivariant cohomology class with non-constant coefficients in the $(n-2)$-dimensional homology group of an $(n-2)$-sphere, the "obstruction class", vanishes. Does it?

A picture show

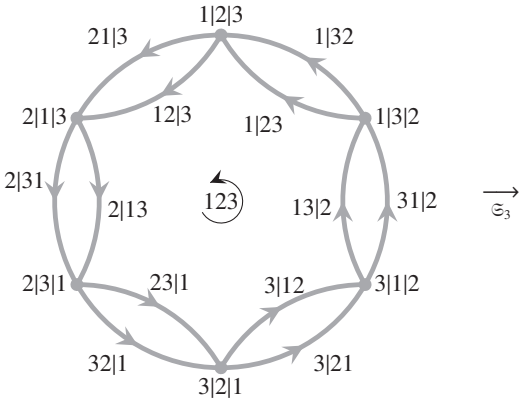
For $n = 3$, we have to find out whether there is an $\mathfrak{S}_3$-equivariant map $\mathcal{F}(2, 3) \longrightarrow S^1$. The space $\mathcal{F}(2, 3)$ is a cell complex with 6 vertices, 12 edges and 6 hexagon 2-faces. Only one of the hexagons is shown in our figure; however, one hexagon is as good as all of them, as an equivariant map is specified by its image on one of them.

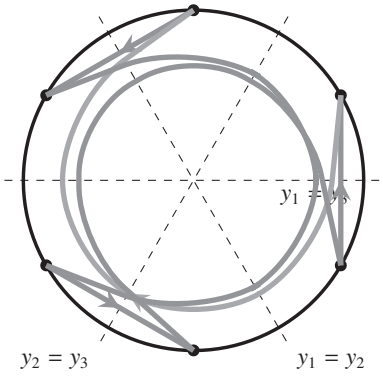
We have to map the 1-skeleton (graph) of the cell complex, in particular the boundary of the hexagon, in such a way that the map can be extended to the interior, as a map to the circle S^1, which does *not* have an interior. The "obvious" equivariant map takes the boundary of the hexagon to the circle, going around once, as indicated by the six directed edges. This can be interpreted as a map of 1-spheres of degree 1, so it does not extend to the hexagon.





We could now try to modify the map by changing it on one of the six edges of the hexagon. However, as we have to maintain an *equivariant* map, the map has to be changed simultaneously on the whole orbit, that is, on all images of our edge under the action of the permutation group. One can now work out that any change of the map on one of the edges affects two further edges of the hexagon, as indicated in the following picture:





Can we achieve a mapping degree of 0, and thus a map that can be extended to the interior of the hexagon, by such equivariant changes (on triples of edges)? It is not hard to see that this amounts to the following. An equivariant map

$$\mathcal{F}(2,3) \underset{\mathfrak{S}_3}{\longrightarrow} S^1$$

exists if and only if there is a solution to

$$1 + 3x_1 + 3x_2 = 0$$

for integers $x_1, x_2 \in \mathbb{Z}$ – and clearly there is none. So, as the map does *not* exist, the counterexample to the Nandakumar and Ramana Rao conjecture for $n = 3$ does not exist. So we have established the case $n = 3$!

How about $n = 4$? This turns out to be quite a similar story, except now on the left side we have a 3-dimensional cell complex, on $4! = 24$ vertices and 24 maximal cells, which have the combinatorics of a permutahedron. (You can find permutahedra explained, for example, in [16], which on *ebay* you might find classified under "esoterics" . . .)

Thus, here is the image for our situation:

$$\underset{\mathfrak{S}_4}{\longrightarrow} S^2$$

One can easily work out that the 14 faces of the permutahedron come in three separate equivalence classes (orbits) under the action of the symmetric group $\mathfrak{S}_4$: there's the orbit of 6 squares and two orbits of 4 hexagons each. Thus, we find that an equivariant map

$$\mathcal{F}(2,4) \underset{\mathfrak{S}_4}{\longrightarrow} S^2$$

exists if and only if there is a solution to

$$1 + 4x_1 + 6x_2 + 4x_3 = 0$$

for integers $x_1, x_2, x_3 \in \mathbb{Z}$ – but clearly there is none.

You see the pattern? For general n, an equivariant map

$$\mathcal{F}(2,n) \underset{\mathfrak{S}_n}{\longrightarrow} S^{n-2}$$

exists if and only if there is a solution to

$$1 + \binom{n}{1}x_1 + \cdots + \binom{n}{n-1}x_{n-1} = 0$$

for integers $x_1, \ldots, x_{n-1} \in \mathbb{Z}$, that is, if the n-th row of Pascal's triangle, with the 1 at the end removed, does not have a common factor.

The queen

We started this story with a little discrete geometry problem from a 2006 blog post from India. The journey led us to use optimal transport, to devise a setup about equivariant continuous maps, that is, topology, and to compute the obstruction class in equivariant cohomology. This obstruction class does not vanish, that is, an equivariant map does not exist, if there is no solution to a certain Diophantine equation. We have arrived at number theory.

> "Mathematics – according to Gauß in his own words – is the Queen of the Sciences and number theory is the Queen of Mathematics.
> She often condescends to render service to astronomy and other natural sciences but in all relations she is entitled to the first rank."

We know this from Wolfgang Sartorius von Waltershausen, Gauß' friend, who spoke the eulogy at his grave and wrote the first biography. Sartorius von Waltershausen was a geologist and Gauß worked hard on geography ("measuring the world") as well as on astronomy (the rediscovery of Ceres made him famous in 1801 and not the *Disquisitiones Arithmeticae*, which no one understood at the time). So, Gauß' statement about number theory as the Queen of Mathematics and mathematics as the Queen of the Sciences is authentic and it carries weight.

And as so often happens, for our problem, the final punch line also belongs to number theory. You can easily work out your Pascal's triangle; this is child's play. You probably get stuck at the sixth row – as in the figure on the cover (taken from the children's book *The Number Devil* by the German writer Hans Magnus Enzensberger).

Indeed, for $n = 6$, we get the equation

$$1 + 6x_1 + 15x_2 + 20x_3 + 15x_4 + 6x_5 = 0,$$

which *does* have an integer solution (indeed, $x_1 = x_2 = -1$, $x_3 = 1$ and $x_4 = x_5 = 0$ will do), so the map *does* exist . . .

So what is special about 6? And how about general n? Our quest takes us back to India but roughly 100 years earlier. At the beginning of the first volume of the Journal of the Mathematics Club of Madras (this is the time and the place where Ramanujan came from!), Balak Ram published the following result:

Theorem (Balak Ram [15]). *The equation*

$$1 + x_1\tbinom{n}{1} + \cdots + x_{n-1}\tbinom{n}{n-1} = 0$$

has no solution in integers $x_1, \ldots, x_{n-1} \in \mathbb{Z}$, that is, the interior of the n-th row of Pascal's triangle has a common factor, if and only if n is a prime power.

Thus, the Nanadkumar and Ramana Rao problem is solved for the case when n is a prime power but it remains open for general n, for now.

Theorem (Blagojević & Z. [7]). *If n is a prime power, then every polygon admits a "fair" partition into n parts.*

In all other cases, for $n = 6, 10, 12, \ldots$, the problem remains open – although, in [7], we state a much more general theorem which happens to be true *if and only if* n is a prime power. Again, this is derived from the result of our EOT computation: *there is an equivariant continuous map*

$$F(\mathbb{R}^2, n) \underset{\mathfrak{S}_n}{\longrightarrow} S^{n-2}$$

if and only if n is not a prime power.

Three remarks (on proofs) before I stop

First remark (on proofs in public)

According to Victor Klee (1925–2007):

> Proofs should be communicated only between consenting adults in private.

What has been presented here, of course, is not a proof but only a sketch. Details are to be filled in. Details are important.

Second remark (on simple/pretty proofs)

You may know Paul Erdős' story about THE BOOK, which God maintains and which contains the pretty proofs, the perfect proofs, the perfectly simple proofs, the *BOOK proofs* of mathematical theorems. Erdős also liked to say that, as a mathematician, you do not have to believe in God but you should believe in THE BOOK. (Incidentally, this part of Erdős' quote is missing from the 2001 Farsi translation of [1].)

On the other hand, not everyone agrees with the first part, either. Solomon Lefschetz (1884–1972), the demi-god of Princeton mathematics, is quoted as saying:

> Don't come to me with your pretty proofs. We don't bother with that baby stuff around here.

... in particular, when his own students came up with simpler, or more complete, proofs of his results.

Third remark (on correct proofs)

About the same Solomon Lefschetz, it was said that:

> He never wrote a correct proof
> or stated an incorrect theorem.

Apparently, there was a lot of truth in this as well. Some of that was inevitable. Lefschetz was a pioneer in the use of topological methods in algebraic geometry. He described this later:

> As I see it at last it was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry.

However, he did this at a time where the secure mathematical foundations of algebraic topology had not been established – so there was risk and fun in the use of topological methods at the time. Today, there is less risk but not less fun, I believe.

Fourth remark (on correct proofs, II)

In doing mathematics, we are guided by intuition and often "believe" more than we can rigorously establish. This leads to statements of the following form:

> We prove the answer to be yes for $n = 4$ and also discuss higher powers of 2.

This is from the final (sixth) preprint version arXiv:0812.2241v6 of Nandakumar & Ramana Rao's paper [14]. The published version then states:

> We give an elementary proof that the answer is yes for $n = 4$ and generalize it to higher powers of 2.

Indeed, [14] contains beautiful ideas but the proofs given for $n = 4$ and sketched for higher powers of 2 need further work.[2] On the other hand, Karasev, Hubard & Aronov give a different argument for the prime power case of the Nandakumar and Ramana Rao problem, published in *Geometriae Dedicata* in 2014 [11]. There seems to be no way to make their approach, which would need a relationship between the Euler class and homology classes induced by generic cross-sections, rigorous and complete.[3] Karasev summarises this on his homepage as follows:

> In this version we make a clearer and slightly more general statement of the main theorems and spend some effort to explain the proof of the topological lemma. Our approach to the topological lemma is the same that D. B. Fuks and V. A. Vasiliev used to establish its particular cases. Another, more technical and more rigorous, approach to the topological lemma can be found in the paper arXiv:1202.5504 of P. Blagojevic and G. Ziegler.

Fifth remark (on counting)

I promised three remarks. On the other hand, we all know that there are three kinds of mathematician: those who can count and those who can't.

A poem before I stop

This paper was to demonstrate how a little "sparrow problem" serves as a test instance for some "big theory cannons". However, there are many sparrow problems around, such as a multiple incidence problem known as the (coloured) Tverberg problem (see, for example, [17] and [4]) or the existence of highly regular maps $\mathbb{R}^d \to \mathbb{R}^N$, continuous maps that are required to map any k distinct points to linearly independent vectors [6]. Progress on these problems not only "uses" intricate topological theory but it depends on progress in the understanding and computation of subtle algebraic topology information about configuration spaces and on the development of advanced theory. Thus, we also make progress within

2 For example, [14, Lemma 4] is not true as stated, as one can see in the special case of a square, where for a vertical bisector of the square the resulting rectangles have fair partitions into two convex parts with a continuous range of perimeters but for a nearby bisector of the square the resulting quadrilaterals have only finitely many fair partitions.

3 Karasev et al. also set out to show the non-existence of the equivariant map $F(\mathbb{R}^d, n) \to S^{n-2}$ in the case when n is a prime power. For this, they intend to show that the Euler class with twisted coefficients of a natural vector bundle over the open manifold $F(\mathbb{R}^d, n)/\mathfrak{S}_n$ is non-zero. However, the relationship between a generic cross-section and the Euler class of the bundle via Poincaré duality breaks down over open manifolds. For a more detailed discussion, see [7, p. 51].

algebraic topology, which has allowed us to solve longstanding technical problems, such as the extended Vassiliev conjecture [3].

So the relationship between cannons and sparrows is quite a bit more complex than one might have thought. We *do* shoot with cannons at sparrows and, on some occasions, the sparrows fight back. Let the poet speak:

Two errors

I must admit that on occasion
I have shot sparrows at cannons.

There was no bull's eye in that,
which I understand.

[. . .]

Cannons against sparrows,
that would be to lapse into the inverse error.

Hans Magnus Enzensberger

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Photo: Sandro Most

Günter M. Ziegler [gmziegler@zedat.fu-berlin.de] is a professor of mathematics at Freie Universität Berlin. His current work connects combinatorics and discrete geometry (especially polytopes) with topology (in particular configuration spaces). His honours include a Leibniz Prize (2001), the MAA Chauvenet Prize (2004) and the 2008 Communicator Award, as well as an ERC Advanced Grant (2010–2015). He presented invited lectures at the ICMs in Beijing (2002), Hyderabad (2016) and Seoul (2014). From 2006 to 2008, he was President of the German Mathematical Society. He is a member of the Executive Board of the Berlin-Brandenburg Academy of Sciences and a member of the Senate of the German Science Foundation.

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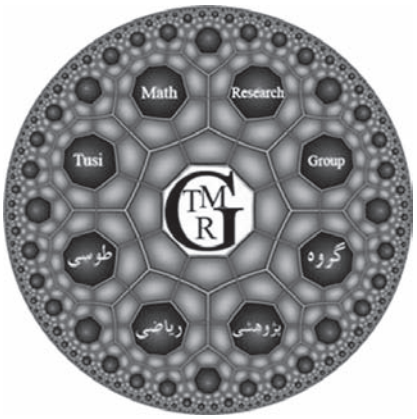

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Weierstrass and Uniform Approximation

Joan Cerdà (Universitat de Barcelona, Spain)

> *… il y avait tout un type de raisonnements qui se ressemblaient tous et qu'on retrouvait partout; ils étaint parfaitement rigoureux, mais ils étaient longs. Un jour on a imaginé le mot d'uniformité de la convergence et ce mot seul les a rendus inutiles …* (Poincaré in "Science et Méthode")

1 The Weierstrass rigour, analytic functions and uniform convergence

Two subjects are central to the work of Karl Weierstrass: his programme of basing analysis on a firm foundation starting from a precise construction of real numbers, known as arithmetisation of analysis, and his work on power series and analytic functions. Concerning the former, in a 1875 letter to H. A. Schwarz where he criticised Riemann, Weierstrass states:

> Je mehr ich über die Prinzipien der Funktionentheorie nachdenke, und ich thue dies unablässig, um so fester wird meine Überzeugung, dass diese auf dem Fundament einfacher algebraischer Wahrheiten aufgebaut werden müssen.
> [The more I think about the principles of function theory, and I do this without cease, the firmer becomes my conviction that this must be based on the foundation of algebraic truths.]

Concerning the topic of series, Weierstrass said that his work in analysis was nothing other than power series. He chose power series because of his conviction that it was necessary to construct the theory of analytic functions on simple "algebraic truths". He started the use of analytic functions when he became interested in Abel's claim that elliptic functions should be quotients of convergent power series. A general view of his work on analytic functions is included in [Bi]. His work relating to the uniform approximation of functions by polynomials is a clear example of both subjects.

In 1839, Weierstrass was in Münster, where he attended Christoph Gudermann's lectures on elliptic functions. It is in an 1838 article about these functions by Gudermann where the concept of uniform convergence probably appears for the first time. There he talks about "convergence in a uniform way", where the "mode of convergence" does not depend on the values of the variables. He refers to this convergence as a "remarkable fact" but without giving a formal definition or using it in proofs. But it was Weierstrass, in a work in Münster dated 1841 but only published in 1894, who introduced the term "uniform convergence" and used it with precision.

The (pointwise) convergence of a sequence of functions was used in a more or less conscious way from the beginning of infinitesimal calculus, and the continuity of a function was clearly defined by Bolzano and Cauchy. Cauchy did not notice uniform convergence and in 1821 he believed he had proved the continuity of limits and sums of series of continuous functions. Dirichlet found a mistake in Cauchy's proof and Fourier and Abel presented counter-examples using trigonometric series. Abel then proved the continuity of the sum of a power series by means of an argument using the uniform convergence in that special case. This convergence was also considered in 1847, by Seidel in a critique to Cauchy (talking without a definition about slow convergence as the absence of uniform convergence) and, independently, by Stokes in 1849 (talking about infinitely slow convergence, with no reference to Cauchy) but without any impact on further development. G. H. Hardy, when comparing the definitions by Weierstrass, Stokes and Seidel in the paper "Sir George Stokes and the concept of uniform convergence", remarked that "Weierstrass's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis".

At the end of the 19th century, under the influence of Weierstrass and Riemann, the application of the notion of uniform convergence was developed by many authors such as Hankel and du Bois-Raymond in Germany, and Dini and Arzelà in Italy. For instance, Ulisse Dini proved the theorem, nowadays named after him, that if an increasing sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ is pointwise convergent, the convergence is uniform. Uniform convergence was used by Weierstrass in his famous pathological continuous function with no derivative at any point

$$f(\vartheta) = \sum_{n=0}^{\infty} b^n \cos(a^n \vartheta), \tag{1}$$

which he discovered in 1862 (and published in 1872). This was a counter-example to the general belief that continuous functions had to be differentiable except at some special points, showing that the uniform limit of differentiable functions need not be differentiable. This was also Riemann's conviction, who stated without proof in 1861 that

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} \tag{2}$$

was an example of a nowhere differentiable continuous function. Weierstrass remarked that "it is somewhat difficult to check that it has this property". Much later, in 1916, Hardy proved that this function has no derivative at any point except at rational multiples of π (for rational numbers that can be written in reduced form as p/q with p and q odd integers).

Weierstrass observed that Riemann apparently believed that an analytic function can be continued along any curve that avoids critical points. He remarked that this is not possible in general, as shown by the lacunar series $u(z) = \sum_{n=0}^{\infty} b^n z^{a^n}$, where a is an odd integer and $0 < b < 1$ such

that $ab > 1 + 3\pi/2$. In this case, the circle $|z| = 1$ is the natural boundary for the function u. Indeed, the convergence radius of the series is 1 and the real part of the sum on $z = e^{i\vartheta}$ is the pathological function (1).

In the 1830s, Bolzano constructed a continuous nowhere differentiable function that was the uniform limit of certain piecewise linear functions but with the usual mistake of assuming that pointwise limits of continuous functions are also continuous. He only stated that his function was not differentiable in a dense set of points but, in fact, the function is everywhere non-differentiable.

M. Kline [Kl] explains that in 1893 Hermite said to Stieltjes in a letter: "Je recule de terreur et d'aversion devant ce mal déplorable que constituent les fonctions continues sans dérivées", and this was a very widespread opinion. In this sense, talking about teaching mathematics, Poincaré says in [P1] and in [P2]:

> La logique parfois engendre des monstres. Depuis un demi-siècle on a vu surgir une foule de fonctions bizarres qui semblent s'efforcer de ressembler aussi peu que possible aux honnêtes fonctions qui servent à quelque chose. Plus de continuité, ou bien de la continuité, mais pas de dérivées, etc. Bien plus, au point de vue logique, ce sont ces fonctions étranges qui sont les plus générales, celles qu'on rencontre sans les avoir cherchées n'apparaissent plus que comme un cas particulier. Il ne leur reste qu'un tout petit coin. Autrefois, quand on inventait une fonction nouvelle, c'était en vue de quelque but pratique; aujourd'hui, on les invente tout exprès pour mettre en défaut les raisonnements de nos pères, et on n'en tirera jamais que cela.

Of course, we are now familiar with continuous nowhere differentiable functions and they are needed to deal with fractals, Brownian motion, wavelets, chaos, etc., but these examples were a shock for the mathematical community of the 19th century.

In Section 2, we are concerned with the Weierstrass proof of his approximation theorem, that is, the opposite of the problem considered above: every continuous function is the uniform limit of differentiable functions, in fact the uniform limit of polynomials.

2 1885: The Weierstrass approximation theorem

Weierstrass' 1885 article [W], which is divided into two parts, has to be included in his programme of representing functions by means of power series. There, he proved the following approximation theorem at the age of 70.

Theorem 1. If $f \in C[a, b]$ is a continuous real function on the closed interval $[a, b]$ and $\varepsilon > 0$ then $|f - p| \leq \varepsilon$ for some polynomial p.

The importance of this fact was immediately appreciated and one year later a translation, also in two parts, was included in the Journal de Liouville. Many of the best mathematicians became strongly interested in this result and gave new demonstrations and new applications. Among them were Runge, Lerch and Mittag-Leffler, Weierstrass' students, and many others, such as Picard, Fejér, Landau, de la Vallée Poussin, Phragmén, Lebesgue, Volterra, Borel and Bernstein (see, for example, [Pi]).

In order to describe the Weierstrass paper, and several of the proofs that appeared later, we can suppose that $[a, b] = [0, 1]$, $f(0) = f(1) = 0$ and $|f| \leq 1$ without loss of generality. The basic elements of Theorem 1 are included in the first part of [W], namely in Theorems (A), (B) and (C). Weierstrass considered the function

$$F(x,t) = \frac{1}{t\sqrt{\pi}} \int_{-\infty}^{\infty} f(u) e^{-\left(\frac{u-x}{t}\right)^2} du$$

similar to the solution of the heat equation $\partial_t F(x,t) = \partial_x^2 F(x,t)$ on the half plane $t > 0$ with initial values $F(x,0) = f(x)$ given by Fourier and Poisson.

The proof is divided into two parts: first it is shown that $F(x,t) \to f(x)$ uniformly on every interval of $\mathbb{R}$ as $t \downarrow 0$ and then that, for a given $t > 0$, $F(\cdot, t)$ is an entire function that can be approximated by polynomials using the partial sums of the Laurent series, as follows.

Theorem (A) *Let f be a [bounded] continuous function on $\mathbb{R}$. Then there are many ways (*auf mannigfaltige Weise*, in Weierstrass' words) to choose a family of entire functions $F(x,t)$, with $t > 0$ a real parameter, such that $\lim_{t\to 0} F(x,t) = f(x)$ for every $x \in \mathbb{R}$.*

Moreover, the convergence is uniform on every interval $[x_1, x_2]$.

Weierstrass uses the convolution

$$F(x,t) = \frac{1}{t} \int_{-\infty}^{\infty} f(u)\psi\left(\frac{u-x}{t}\right) du,$$

where ψ is a general kernel satisfying the conditions:

1. ψ is similar to f (bounded and continuous).
2. $\psi \geq 0$ and $\psi(-x) = \psi(x)$.
3. The improper integral $\int_{-\infty}^{\infty} \psi(x)\,dx$ is convergent (Weierstrass divides by $\omega = \int_{-\infty}^{\infty} \psi(x)\,dx$ and one can suppose that $\int_{-\infty}^{\infty} \psi(x)\,dx = 1$).

In a first step, by Cauchy's criterion, he shows that, for every x, the integrals $\int_{-a}^{b} f(u)\psi(\frac{u-x}{t})\,du$ converge as $a, b \to \infty$ using the average property of integrals and the properties of ψ.

$$\begin{aligned}\Delta &:= \frac{1}{t}\int_{-b_1}^{b_2} f(u)\psi\left(\frac{u-x}{t}\right)du - \frac{1}{t}\int_{-a_1}^{a_2} f(u)\psi\left(\frac{u-x}{t}\right)du \\ &= \frac{1}{t}\int_{-b_1}^{-a_1} f(u)\psi\left(\frac{u-x}{t}\right)du - \frac{1}{t}\int_{-a_2}^{b_2} f(u)\psi\left(\frac{u-x}{t}\right)du \\ &= f(\xi_1)\int_{(a_1+x)/t}^{(b_1+x)/t} \psi(v)\,dv + f(\xi_2)\int_{(a_2-x)/t}^{(b_2-x)/t} \psi(v)\,dv\end{aligned}$$

and, if $b_1 > a_1 \to \infty$ and $b_2 > a_2 \to \infty$ then

$$|\Delta| \leq M\int_{(a_1+x)/t}^{(b_1+x)/t} \psi(v)\,dv + M\int_{(a_2-x)/t}^{(b_2-x)/t} \psi(v)\,dv \to 0.$$

Now, as we currently do with summability kernels, he decomposes F:

$$\begin{aligned}F(x,t) &= \frac{1}{t}\left\{\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{+\infty} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta}\right\} f(u)\psi\left(\frac{u-x}{t}\right)du \\ &= (f(\xi_1) + f(\xi_2))\int_{\delta/t}^{\infty} \psi(u)\,du \\ &= \int_0^{\delta/t} [f(x-tu) + f(x+tu)]\psi(u)\,du\end{aligned}$$

and $F(x,t) - f(x)$ is equal to

$$(f(\xi_1) - f(x)) \int_{\delta/t}^{\infty} \psi(u)\,du + (f(\xi_2) - f(x)) \int_{\delta/t}^{\infty} \psi(u)\,du + \int_0^{\delta/t} [f(x-tu) + f(x+tu) - 2f(x)]\psi(u)\,du.$$

Taking the absolute values and using the uniform continuity of f on every interval $[x_1, x_2]$, it follows that the last integral is bounded by any $\varepsilon > 0$ if δ is small and $t \le t_0$, for some $t_0 > 0$. The first two terms of the right side are bounded by $\int_{\delta/t}^{\infty} \psi(u)\,du$, which is also $\le \varepsilon$ if $t \downarrow 0$, for any δ.

Weierstrass explicitly observes that the convergence is uniform on $[x_1, x_2]$.

Theorem (B). *If f is as in Theorem (A) and $\varepsilon > 0$ then there are many ways of choosing a polynomial G, such as $|f(x) - G(x)| \le \varepsilon$ for every $x \in [x_1, x_2]$.*

In the proof, he observes that for many kernels, as in the case of the heat kernel $W(x) = (1/\pi)e^{-x^2}$ (which we call the Weierstrass kernel), every $F(\cdot, t)$ is entire and can be represented as

$$F(x,t) = \sum_{n=0}^{\infty} A_n x^n.$$

Hence, for any $\varepsilon > 0$, there is a polynomial $G_N(x) = \sum_{n=0}^{N} A_n x^n$ that satisfies

$$|F(x,t) - G_N(x)| \le \varepsilon \quad (x \in [x_1, x_2]),$$

that is, $|f - G_N| \le 2\varepsilon$.

We have seen the proof of the approximation but Weierstrass wanted to obtain a representation of f as a series. Therefore, he presented a third theorem:

Theorem (C) *If $f(x)$ is as before, it can be "represented" in many ways as [the sum of] a series of polynomials that is uniformly convergent on every interval and absolutely convergent for every x.*

The proof is easy. He writes $\varepsilon = \sum \varepsilon_n$ and approximates f as in Theorem (B) by Taylor polynomials G_n so that $|f - G_n| < \varepsilon_n$ on $[-a_n, a_n]$, with $a_n \uparrow \infty$. Finally, the telescoping series $G_1 + \sum_{n=1}^{\infty}(G_{n+1} - G_n)$ has the required properties.

In the case of a continuous function f on an interval $[a, b]$, Weierstrass extended f to $\mathbb{R}$ defining $f(x) = f(a)$ if $x < a$ and $f(x) = f(b)$ if $x > b$. The sequence $\{G_n\}$ of polynomials tends to f uniformly on $[a, b]$.

In the second part of [W], even though du Bois-Reymond had constructed a continuous function with Fourier series not converging to the given function in a dense set of points, Weierstrass, using methods of complex analysis, proves that any periodic function is the uniform limit of trigonometric polynomials:

Theorem 2. Let f belong to the set $C_{2\pi}(\mathbb{R})$ of continuous 2π-periodic functions on $\mathbb{R}$ and let $\varepsilon > 0$. Then there are trigonometric polynomials t such that $|f - t| \le \varepsilon$ on $\mathbb{R}$.

The proof goes as follows. If $\psi(z) = (1/\pi)e^{-z^2}$ (or similar) then, for every $t > 0$,

$$F_t(z) := \frac{1}{t} \int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-z}{t}\right) du$$

is an entire periodical function, so

$$G_t(z) := F_t\left(\frac{\log z}{i}\right)$$

is a univalent analytic function, well defined on $\mathbb{C} \setminus \{0\}$, which is real valued on $\mathbb{R}$ and has a Laurent series $G_t(z) = \sum_{k=-\infty}^{+\infty} c_{t,k} z^n$ that on the unit disc $z = e^{ix}$ is

$$F_t(x) = \sum_{k=-\infty}^{+\infty} c_{t,k} e^{ikx},$$

with uniform convergence.

A "double ε-argument" completes the proof: if $t > 0$ is small enough and N large enough then

$$|f(x) - F_t(x)| \le \varepsilon, \quad \left|F_t(x) - \sum_{k=-N}^{+N} c_{t,k} e^{ikx}\right| \le \varepsilon \qquad (\forall\, x).$$

In fact, we can suppose that $\sum_{k=-N}^{+N} c_{t,k} e^{ikx}$ is real, since

$$\left|F_t(x) - \Re \sum_{k=-N}^{+N} c_{t,k} e^{ikx}\right| \le \left|F_t(x) - \sum_{k=-N}^{+N} c_{t,k} e^{ikx}\right|.$$

Weierstrass observes that this result justifies the solution of the heat equation given by Fourier for a thin annulus with a given initial temperature, since his trigonometric polynomials also satisfy the heat equation.

3 1891: Picard and the approximation obtained from the Poisson equation

Émile Picard (1856–1941), who in his famous "Traité d'analyse" included several proofs of the Weierstrass theorem without citing the original reference, obtained in 1891 (see [Pc]) a new proof similar to the one given by Weierstrass. He replaced the heat kernel by the Poisson kernel, which is also used to solve the Dirichlet problem corresponding to a stationary distribution of the temperature $u(r, \vartheta)$ on the disc $D = \{(r, \vartheta);\ 0 \le r < 1\}$ for a continuous and periodic function $f(\vartheta) = u(r, \vartheta)$ representing the temperature on the boundary. If $z = x + iy = re^{i\vartheta} \in D$ then the solution is given by

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1 - r^2}{1 - 2r\cos(\vartheta - t) + r^2}\, dt = \Re \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t)\, dt,$$

which is harmonic, as the real part of a holomorphic function, that is, $\triangle u(x, y) = 0$.

On the other hand, if $0 < r < 1$, the summation of a geometric series gives

$$\frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\vartheta - t) + r^2} = P_r(\vartheta - t)$$

with

$$P_r(s) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{iks}.$$

The family of periodic functions $\{P_r\}_{0<r<1}$, currently called the Poisson kernel, satisfies:

1. $P_r \ge 0$;
2. $P_r(-s) = P_r(s)$;
3. $\int_{-\pi}^{\pi} P_r(s)\, ds = 1$; and
4. $\sup_{0<\delta\le|t|\le\pi} P_r(t) \le P_r(\delta) \to 0$ as $\delta \downarrow 0$.

Consequently, as with the Weierstrass kernel,

$$u(re^{i\vartheta}) = \int_{-\pi}^{\pi} f(t)P_r(\vartheta - t)\,dt \to f(\vartheta)$$

uniformly as $r \uparrow 1$. If $0 < r < 1$ then

$$u(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} r^{|k|}e^{ik(\vartheta-t)}f(t)\,dt = \sum_{k=-\infty}^{\infty} c_k r^{|k|}e^{ik\vartheta}$$

with

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt$$

and the series is uniformly convergent, by the Weierstrass M-test.

If $f : [0,1] \to \mathbb{R}$ is a continuous function such that $|f| \le 1$ and $f(0) = f(1) = 0$ then it is extended by zero to $[-\pi, \pi]$ and to a periodic function to $\mathbb{R}$. For every $\varepsilon > 0$, there is some $0 < r < 1$ such that

$$\sup_{-\pi\le\vartheta\le\pi} |f(\vartheta) - u(re^{i\vartheta})| \le \varepsilon.$$

Notice that

$$\sum_{|k|\ge N}^{\infty} |c_k r^{|k|}e^{ik\vartheta}| \le \sum_{|k|\ge N}^{\infty} r^{|k|} = 2\frac{r^N}{1-r}$$

and one can choose N so that $2r^N/(1-r) < \varepsilon$. By adding these inequalities,

$$\sup_{-\pi\le\vartheta\le\pi} \Big|f(\vartheta) - \sum_{|k|<N} c_k r^{|k|}e^{ik\vartheta}\Big| \le \sup_{-\pi\le\vartheta\le\pi} |f(\vartheta) - u(re^{i\vartheta})| + 2\frac{r^N}{1-r} \le 2\varepsilon.$$

In this way, Picard obtains a trigonometric polynomial $Q(\vartheta)$ that uniformly approximates f and, in turn, Q is approximated by a Taylor polynomial. This is the proof presented, for instance, in [Se]. At the end of his paper, Picard observes that the same method gives the approximation theorem for functions of several variables. He also says that his proof is based on an inequality due to H. Schwarz; in fact, in [Sc] we almost find the proof.

When the Weierstrass Mathematische Werke were re-edited in 1903, the same remark about the case of several variables was included.

Instead of this last approximation by a Taylor polynomial, similar to the one given by Weierstrass, in 1918 and following an idea by Bernstein, de la Vallée Poussin gave a new, more direct approximation. For $f \in C[-1,1]$ such that $f(-1) = f(1)$, the even function

$$g(\vartheta) := f(\cos\vartheta) \qquad (|\vartheta| \le \pi)$$

is approximated by a 2π-periodic trigonometric polynomial t, $|g - t| \le \varepsilon$, which in turn decomposes into its even and odd parts, $t = t_e + t_o$ ($t_e(\vartheta) = (t(\vartheta) + t(-\vartheta))/2 = \sum_{k=0}^{N} a_k \cos(km)$). Since g is even, it follows that $|g - t_e| \le \varepsilon$. Every $\cos(k\vartheta)$ is a polynomial of $\cos\vartheta$, $\cos(k\vartheta) = T_k(\cos\vartheta)$ (T_k is a Txebisxef polynomial) and $p(x) = \sum_{k=0}^{N} a_k T_k(x)$ satisfies $|f - p| \le \varepsilon$.

4 1898: Lebesgue and polygonal approximations

Other new proofs of the Weierstrass theorem are based on an approximation of the continuous function $f : [0,1] \to \mathbb{R}$ by a polygonal function g with nodes $0 = x_0 < x_1 < \cdots < x_m = 1$,

$$g(x) = g_1(x) + [g_2(x) - g_1(x)]\chi(x - x_1) + \cdots + [g_m(x) - g_{m-1}(x)]\chi(x - x_{m-1}), \qquad (3)$$

where $\chi(x) = 1$ if $x \ge 0$ and $\chi(x) = 0$ if $x < 0$ and where g_j is the line connecting $(x_{j-1}, f(x_{j-1}))$ with $(x_j, f(x_j))$. If we denote $x^+ = \max(x, 0) = (|x| + x)/2$ and $g_1(x) = cx + c_0$ then

$$g(x) = cx + c_0 + c_1(x - x_1)^+ + \cdots + c_{m-1}(x - x_{m-1})^+$$

or, since $(x - x_j)^+ = (|x - x_j| + x - x_j)/2$,

$$g(x) = ax + b_0 + b_1|x - x_1| + \cdots + b_{m-1}|x - x_{m-1}|. \qquad (4)$$

This is how Lebesgue, in his first paper [L1], written when he was 23 years old, gives one of the most elegant proofs of Theorem 1, with the polygonal function g written as in (4). He observed that an approximation of $|x|$ by a polynomial p was sufficient, since if

$$\Big||x| - p(x)\Big| \le \varepsilon \quad (|x| \le 1)$$

then, for every $x \in [0,1] \cap [x_k - 1, x_k + 1]$ ($k = 1, \ldots, m-1$), we have

$$\Big||x - x_k| - p(x - x_k)\Big| \le \varepsilon$$

and an approximation of g by polynomials easily follows. To obtain $p(x)$, Lebesgue wrote

$$|x| = \sqrt{1 - (1 - x^2)} = \sqrt{1 - z},$$

with $z = 1 - x^2$, and then

$$\sqrt{1 - z} = \sum_{n=0}^{\infty} C_{1/2}^n (-z)^n,$$

using the binomial formula with

$$C_{1/2}^n = \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - n + 1)}{n!}.$$

The Stirling formula yields that the convergence radius is 1 and for $|z| = 1$ the convenient convergence holds.

The delicate point in the Lebesgue proof was precisely the discussion of the convergence of the series of $\sqrt{1 - z}$ at $|z| = 1$. A simple way to overcome this difficulty is to use that $(1 - \delta z)^{1/2}$ tends to $(1 - z)^{1/2}$ uniformly as $\delta \uparrow 1$ and then it is easy to approximate $(1 - \delta z)^{1/2}$ ($0 < \delta < 1$) by polynomials, since

$$(1 - \delta z)^{1/2} = 1 - \sum_{n=1}^{\infty} a_n \delta^n z^n$$

uniformly on $|z| < \delta^{-1}$, and $\delta^{-1} > 1$.

But probably the most clever way to approximate $|x|$ on $[-1, 1]$ by polynomials was obtained in 1949 by N. Bourbaki [Bo], who recursively defined a sequence of polynomials p_n starting from $p_0 = 0$ and then

$$p_{n+1}(t) = p_n(t) + \frac{1}{2}(t - p_n^2(t)).$$

Since $\sqrt{t} - p_{n+1}(t) = (\sqrt{t} - p_n(t))(1 - \frac{1}{2}(\sqrt{t} + p_n(t)))$, it follows by induction that $0 \le p_n(t) \le \sqrt{t}$, and $\{p_n\}$ is increasing on $[0, 1]$. By Dini's theorem, $p_n \to h$ uniformly, with $h \ge 0$ such that

$$h(t) = h(t) - \frac{1}{2}(t - h^2(t)),$$

that is, $h(t) = \sqrt{t}$. Hence $q_n(x) = p_n(x^2) \to |x|$ uniformly on $[-1, 1]$. In fact, this approximation of $|x|$ is useful to prove

the Stone–Weierstrass theorem, an important extension of the Weierstrass theorem. See, for example, [Bo] or [Di].

In 1908, in a letter to Landau referring to the proofs of Weierstrass and Picard, and to those of Fejér and Landau that we present below, Lebesgue observed that they must be considered in a general setting of convolutions with sequences of nonnegative kernels that are approximations of the identity, a remark that he developed in [L2].

Note that, already in 1892, in an unnoticed paper in Czech, M. Lerch [Le] had also proved Theorem 1 using an approximation of the polygonal function g by a Fourier series of cosine

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\pi x); \quad A_n = 2\int_0^1 g(t)\cos(n\pi t)\,dt.$$

The method of Dirichlet (1829) on the Fourier series says that

$$f \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(2\pi nx) + B_n \sin(2\pi nx)).$$

One concludes by writing the Fourier sums as integrals:

$$s_N(f,x) = \frac{A_0}{2} + \sum_{n=1}^{N} (A_n \cos(2\pi nx) + B_n \sin(2\pi nx)) \qquad (5)$$

$$= \int_{-1/2}^{1/2} f(t) D_N(x-t)\,dt, \qquad (6)$$

where

$$D_N(t) = 1 + 2\sum_{n=1}^{N} \cos(2\pi nt). \qquad (7)$$

As proved by Heine in 1870 [He], $s_N(f,x) \to f(x)$ uniformly if f is continuous and piecewise monotone or if it is piecewise differentiable, as in the case of g. This is the proof contained in [CJ].

In 1897, V. Volterra (1856–1927) presented a similar proof of Theorem 2.

5 1901: Runge's phenomenon

To obtain a direct proof of Theorem 1, it is natural to try a substitution of the polygonal approximation of our continuous function f by an interpolation of the values of $(x_j, f(x_j))$ by polynomials with increasing degrees, when the number of nodes x_j is also increasing. But Carl Runge (1856 Bremen – 1927 Göttingen), in 1901, showed in [R3] that going to higher degrees does not always improve accuracy, like with the Gibbs phenomenon in Fourier series approximations.

Runge observed that if the function

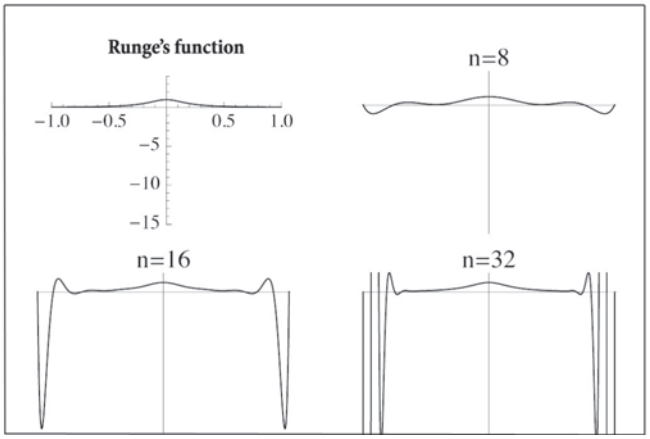
$$f(x) = \frac{1}{1+25x^2}$$

on the interval $[-1,1]$ is interpolated at $n+1$ equidistant nodes x_i,

$$x_j = -1 + (j-1)\frac{2}{n} \qquad (j = 1,2,\ldots,n+1),$$

by a polynomial p_n of degree $\le n$, the resulting interpolation oscillates toward the extremes of the interval and the error tends toward infinity when the degree of the polynomial increases, that is,

$$\lim_{n\to\infty} \Big(\max_{-1\le x\le 1} |f(x) - p_n(x)| \Big) = \infty.$$



Runge's example

It is natural to try interpolation with nodes more densely distributed toward the edges of the interval, since then the oscillation decreases. In the case $f : [-1,1] \to \mathbb{R}$, choosing

$$-1 < x_n < \cdots < x_1 < 1, \quad (x_k = \cos((2k-1)\pi/2n)), \quad (8)$$

which are the zeros of the Chebyshev polynomials, the error is minimised and decreases as the degree of the polynomial increases.

Thirteen years later, in 1914, Faber [Fa] would show that, for any fixed triangular infinite matrix of nodes,

$$-1 \le x_{n,n} < \cdots < x_{0,n} \le 1,$$

there is always a continuous function such that the corresponding sequence of interpolating polynomials p_n of degree n ($p_n(x_{nj}) = f(x_{nj})$) is divergent.

Finally, in 1916, Fejér [R1] proved that Theorem 1 can be obtained by an interpolation scheme *à la* Hermite, with nodes (8) and where the interpolation polynomials are the polynomials H_n of degree $\le 2n-1$ such that $H_n(x_k) = f(x_k)$ and $H_n'(x_k) = 0$.

It is worth noting that, in 1885, Runge had also proved the following result ([R1]), closely related to Theorem 1. If K is a compact subset of $\mathbb{C}$, f is a holomorphic function defined on a neighbourhood of K and A is any set containing at least one point of every "hole" (a bounded component) of $\mathbb{C} \setminus K$ then there exists $r_k \to f$ uniformly on K, where every r_k is a rational function with singularities located in A. Hence, if $\mathbb{C} \setminus K$ is connected, every r_k is a polynomial.

In the same year as the proof of Theorem 1 by Weierstrass, Runge [R2] also proved that every polygonal function g, with the representation (3),

$$g(x) = g_1(x) + \sum_{j=1}^{m-1} [g_{j+1}(x) - g_j(x)]\chi(x - x_j),$$

also admits a uniform approximation by rational functions obtained as follows. Given $\delta > 0$, the increasing sequence of functions

$$\psi_n(x) := 1 - \frac{1}{1+(1+x)^{2n}}$$

is decreasing and tends to 0 on $[-1,0)$ and is increasing and tends to 1 on $(0,1]$. By Dini's theorem, it is uniformly convergent to the function χ on $\{\delta \le |x| \le 1\}$.

Every linear function $g_{j+1} - g_j$ vanishes at x_j and it can be seen that

$$[g_{j+1}(x) - g_j(x)]\psi_n(x - x_j) \to [g_{j+1}(x) - g_j(x)]\chi(x - x_j)$$

uniformly on $[0, 1]$ as $n \to \infty$. Hence

$$R_n(x) := g_1(x) + \sum_{j=1}^{m-1} [g_{j+1}(x) - g_j(x)]\,\psi_n(x - x_j)$$

are rational functions such that $R_n \to f$ uniformly on $[0, 1]$.

As mentioned in a footnote of the Mittag-Leffler paper [ML], Phragmén observed with curiosity in 1886, as a 23-year-old, that Runge did not see that he could easily obtain the approximation by polynomials as an application of his theorem.

6 1902: Fejér and approximation by averages of Fourier sums

Lipót Fejér (1880 Pécs – 1959 Budapest), while still a teenager in 1899-1900, Berlin, after his discussions with Hermann Schwarz, proved "Fejér's theorem", which is published in [F1]. He gave a new proof of Theorem 1, in two steps like Weierstrass: (A) The continuous function $f : [0, 1] \to \mathbb{R}$, supposed to be such that $f(0) = f(1) = 0$ and then periodised, can be approximated by the averages of its Fourier sums; and (B) The Fourier sums, which are obviously entire functions, are then approximated using Taylor polynomials.

The basic step (A) is Fejér's theorem which, for our function f, states that the averages

$$\sigma_N(f, x) = \frac{1}{N} \sum_{n=0}^{N-1} s_n(f, x)$$

of the Fourier sums

$$s_N(f, x) = \frac{A_0}{2} + \sum_{n=1}^{N} (A_n \cos(2\pi nx) + B_n \sin(2\pi nx))$$

uniformly approximate f.

As we have already recalled, Dirichlet had proved in 1829 that if f is piecewise monotone then $s_N(f, x) \to f(x)$, where

$$s_N(f, x) = \int_{-1/2}^{1/2} f(t) D_N(x - t)\, dt,$$

with D_N as in (7),

$$D_N(t) = 1 + 2 \sum_{n=1}^{N} \cos(2\pi nt).$$

In 1873, Paul du Bois-Raymond presented his famous counterexample of a continuous function lacking this property.

It was well known, and Fejér observed it, that taking average smooth fluctuation of sequences, the averages of Fourier sums can be written as

$$\sigma_N(f, x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{N} \sum_{n=0}^{N-1} D_n(x - t) f(t) dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) F_N(x - t)\, dt,$$

where the integral kernels F_N are trigonometric polynomials such that

1. $F_N \geq 0$;
2. $F_N(-t) = F_N(t)$;
3. $\int_{-1/2}^{1/2} F_N(t)\, dt = 1$; and
4. $\lim_{N\to\infty} \max_{\delta \leq t \leq 1/2} F_N(t) = 0$ if $0 < \delta < 1/2$.

These are typical properties of approximations of the identity and, as in the case of Weierstrass kernels, give uniform convergence $\int_0^1 f(t) F_N(x - t)\, dt \to f(x)$ if $N \to \infty$. The functions $\int_0^1 f(t) F_N(x - t)\, dt$ are also trigonometric polynomials. This proof, similar to the one given by Lerch, is included, for instance, in [Ap].

As previously mentioned, in 1916, Fejér gave in [F2] another proof of Theorem 1 by interpolation.

7 1908: Landau presents a simple proof

Edmund Landau (1877–1938), mathematical grandson of Weierstrass (Frobenius was his advisor), presented in [La] the most elementary proof of Theorem 1. It is a direct proof in one single step. It has been included in several textbooks, such as Rudin's [Ru]. For the sake of the proof, we can suppose that $f(0) = f(1) = 0$, that f is extended by zero to the whole line and that $|f| \leq 1$.

On $[-1, 1]$, consider the so-called Landau kernel, that is, the even polynomials

$$Q_n(x) = c_n(1 - x^2)^n,$$

with $n \geq 0$ and c_n, so that $\int_{-1}^{1} Q_n = 1$. Notice that Bernoulli's inequality $(1+h)^n \geq 1+nh$, with $h \geq -1$, implies that $c_n < \sqrt{n}$. Every Q_n is extended by zero to a function on $\mathbb{R}$ and then

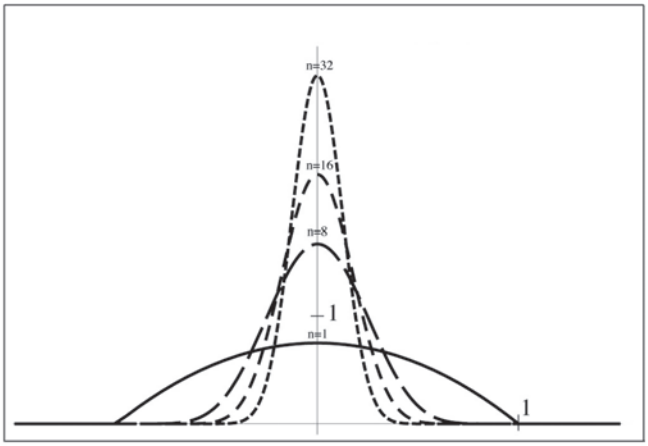
1. $Q_n \geq 0$;
2. $\int_{-\infty}^{\infty} Q_n(x)\, dx = \int_{-1}^{1} Q_n(x)\, dx = 1$; and
3. if $0 < \delta \leq |x| \leq 1$ then $Q_n(x) < \sqrt{n}(1 - x^2)^n \leq \sqrt{n}(1 - \delta^2)^n$, so that $Q_n(x) \to 0$ as $n \to \infty$, uniformly if $|x| \geq \delta$, for every $\delta > 0$.

We are again in the setting of approximations of the identity. On $[0, 1]$, the functions

$$P_n(x) = \int_{-\infty}^{\infty} f(x - t) Q_n(t)\, dt = \int_{-1}^{1} f(x - t) Q_n(t)\, dt$$
$$= \int_{-\infty}^{\infty} f(t) Q_n(x - t)\, dt = \int_{0}^{1} f(t) Q_n(x - t)\, dt$$

are polynomials, since $x - t \in [-1, 1]$ if $x \in [0, 1]$ when $0 \leq t \leq 1$.

Now, the proof follows as usual using the uniform continuity of f, so that $|f(x) - f(y)| < \varepsilon$ if $|x - y| \leq \delta$. For every



The Landau kernel

$0 \le x \le 1$,

$$P_n(x) - f(x) = \int_{-1}^{1} f(x-t)Q_n(t)\,dt - \int_{-1}^{1} f(x)Q_n(t)\,dt.$$

By splitting $\int_{-1}^{1}$ into $\int_{-\delta}^{\delta} + \Big(\int_{\delta}^{1} + \int_{-1}^{-\delta}\Big)$ and using $|f| \le 1$ and the properties (1)÷(3) of Q_n, we conclude that

$$\begin{aligned} |P_n(x) - f(x)| &\le \int_{-1}^{1} |f(x-t) - f(x)|Q_n(t)\,dt \\ &\le \varepsilon \int_{-\delta}^{\delta} Q_n(t)\,dt + 4\int_{\delta}^{1} Q_n(t)\,dt \\ &\le \varepsilon + 4\sqrt{n}(1-\delta^2)^n, \end{aligned}$$

which becomes $\le 2\varepsilon$ if n is large. Hence $\sup_{0\le x\le 1} |P_n(x) - f(x)| \le 2\varepsilon \quad (n \ge N)$.

Also, in 1908, Charles de la Vallée Poussin (who had proved the prime number theorem simultaneously with Hadamard in 1896) obtained Theorem 2 about the approximation of a 2π-periodic function f by trigonometric polynomials using the periodic analogues of the Landau integrals

$$c_n \int_{-\pi}^{+\pi} f(t)\cos^{2n}\Big(\frac{x-t}{2}\Big)\,dt.$$

8 1911: The probabilistic method of Bernstein

Using probabilistic methods, Sergei Bernstein (Odessa 1880 – Moscow 1968) found a very interesting proof of Theorem 1, contained in [Be] and essentially as follows. Let $x \in [0,1]$ and $\{X_n\}$ be a sequence of independent Bernoulli random variables with parameter x, described by a coin with heads with probability x and tails with probability $1-x$. Then, $S_n = X_1 + \cdots + X_n$ has a binomial distribution

$$P\{S_n = k\} = C_n^k x^k (1-x)^{n-k} \quad (k = 0, 1, \ldots, n),$$

since there are C_n^k ways to obtain k heads and $n-k$ tails in n independent trials.

The mean value of $S_n = \sum_{k=0}^{n} k\chi_{\{S_n=k\}}$ is

$$E(S_n) = \int S_n\,dP = \sum_{k=0}^{n} kP\{S_n = k\} = \sum_{k=0}^{n} kC_n^k x^k (1-x)^{n-k}$$

and the weak law of large numbers says that $S_n/n \to x$ in probability. For every $\delta > 0$,

$$P\left\{\left|\frac{S_n}{n} - x\right| \ge \delta\right\} \le \frac{x(1-x)}{\delta^2 n}.$$

Consider now the continuous function $f : [0,1] \to \mathbf{R}$ such that $|f| \le 1$. The average of the composition

$$f(S_n/n) = \sum_{k=0}^{n} f(S_n(k)/n)\chi_{\{S_n=k\}} = \sum_{k=0}^{n} f(k/n)\chi_{\{S_n=k\}}$$

is

$$E\Big(f\Big(\frac{S_n}{n}\Big)\Big) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P\{S_n = k\} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}.$$

These are the Bernstein polynomials $B_n(f,x)$ associated to f; he proved that $B_n f \to f$ as follows. If $\delta > 0$ is such that $|f(x) - f(y)| \le \varepsilon$ when $|x-y| \le \delta$ (curiously he does not explicitly refer to the uniform continuity of f and fixes a value x_0 for x) then

$$\begin{aligned} \big|f(x) - B_n(f,x)\big| &= \big|E(f(x)) - E(f(S_n/n))\big| \\ &\le \int \big|f(x) - f(S_n/n)\big|\,dP = I + J, \end{aligned}$$

with

$$I = \int_{\{|S_n/n - x| \le \delta\}} |f(x) - f(S_n/n)|\,dP \le 2\varepsilon.$$

By the law of large numbers, one also has

$$\begin{aligned} J = \int_{\{|S_n/n - x| > \delta\}} \big|f(x) - f(S_n/n)\big|\,dP &\le 2P(\{|S_n/n - x| > \delta\}) \\ &\le 2\frac{x(1-x)}{\delta^2 n}. \end{aligned}$$

Consequently, if N is large,

$$\sup_{0\le x\le 1} \big|f(x) - B_n(f,x)\big| \le 2\varepsilon + 2\frac{1}{\delta^2 n} \le 4\varepsilon,$$

for every $n \ge N$. Now, we easily prove uniform convergence $B_n f \to f$ without any reference to probabilities (see [Ce]) but the polynomials $B_n f$ were discovered thanks to the Bernstein probabilistic method.

Remark. The construction of the sequence $\{X_n\}$ of independent Bernoulli random variables with parameter x can be obtained as follows. On $\Omega := \{1,0\}$, we define the probability P such that $P(1) = x$ and $P(0) = 1-x$. Then, on

$$\Omega^{\mathbf{N}} = \{1,0\} \times \{1,0\} \times \{1,0\} \times \{1,0\} \times \cdots,$$

we consider the family $\mathcal{E}$ of all the finite unions of sets $\pi_n^{-1}(j)$ $(j = 0,1;\ n = 1,2,3,\ldots)$, with $\pi_n(j_1, j_2, \ldots) = j_n$, and the additive function of sets $Q : \mathcal{E} \to [0,1]$ such that $Q(\pi_n^{-1}(j)) = x$ if $j = 1$. This function is extended in a natural way to a probability on the σ-algebra generated by $\mathcal{E}$. Now we only need to define $X_n(j_1, j_2, \ldots) = j_n$.

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Joan Cerdà [jcerda@ub.edu, http://garf.ub.es/JC/] is a retired full professor of mathematical analysis from the Universitat de Barcelona. He is mainly interested in real and functional analysis.

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José Sebastião e Silva (1914–1972)

Jorge Buescu (Universidade de Lisboa, Portugal), Luis Canto de Loura (Instituto Superior de Engenharia de Lisboa, Portugal), Fernando P. da Costa (Universidade Aberta, Lisboa, Portugal) and Anabela A. Teixeira (Universidade de Lisboa, Portugal)

Introduction

Sebastião e Silva in 1951

José Sebastião e Silva was the foremost Portuguese mathematician of the 20th century. He was a remarkable man and scientist, with multifarious interests, whose influence in Portuguese society was tremendous and is still felt nowadays.

Born on 12 December 1914 in Mértola (a small town in Portugal's deep South), he graduated in mathematics in 1937 from the University of Lisbon. After a period of five years when he was essentially unemployed, barely subsisting with private lessons and occasional teaching duties in private schools, he was hired as a teaching assistant at the Faculty of Sciences of the University of Lisbon in 1942 and, in the following year, he went to Rome with a scholarship from the *Instituto para a Alta Cultura*. He stayed at the *Istituto di Alta Matematica* until December 1946.

Before leaving for Rome, he was an active member of the *Movimento Matemático* (Mathematical Movement), an informal gathering of young Portuguese mathematicians that was very important in the promotion of a deep renewal in mathematical teaching and research in Portugal [2]. The group was active from about 1937 until the late 1940s and, during this time, made an important public intervention. At the end of the 1940s, the repressive backlash by the right-wing, dictatorial regime, in the wake of the rigged 1945 general elections, led to the firing from university teaching positions, the imprisonment and the exile of a large number of its members.

In about one decade, the *Movimento* accomplished several milestones in Portugal's scientific landscape, such as the creation in 1937 of the scientific journal *Portugaliæ Mathematica* (nowadays published by the EMS Publish-

A group of mathematicians of the *Movimento Matemático* at the 1942 visit of Maurice Fréchet (5th from the right) to Lisbon, at the invitation of the Centro de Estudos Matemáticos de Lisboa. Sebastião e Silva is the 4th from the right, next to Frechét.

ing House), the publication, from 1939 onwards, of the *Gazeta de Matemática*, a magazine devoted to mathematical culture at large and, in particular, to what is presently called "raising public awareness of mathematics" (and is still published nowadays by the Portuguese Mathematical Society) and the founding of the Portuguese Mathematical Society (Sebastião e Silva was its member number 10) in 1940.

At the start of the 1940s, several research centres were created as part of the initiatives of the *Instituto para a Alta Cultura* (a governmental body) and the research activities of the *Movimento Matemático* were integrated into these officially supported bodies. Sebastião e Silva was a member of one of these research centres, the *Centro de Estudos Matemáticos de Lisboa*, and it was with a studentship from this *Centro* that he went to Rome in February 1943.

After returning to Portugal, he obtained his PhD at the University of Lisbon in 1949, where he became *1º assistente* (an extinct position nowadays). From 1951 until 1960 he was a full professor at the *Instituto Superior de Agronomia* (Agronomy School of the Technical University of Lisbon) and from this year until his untimely death in 1972 he was a full professor at the Faculty of Sciences of the University of Lisbon.

Sebastião e Silva received the Artur Malheiros Prize of Lisbon's Academy of Sciences in 1956. He became a corresponding member of the academy in 1959 and a permanent member in 1966.

The influence of Sebastião e Silva in the renewal of the mathematics curriculum at the secondary and university levels was extraordinary. He played a leading part in the establishing of a number of research institutions in Lisbon where, with his example of daily research and his role in research supervision and coordination, he inspired a younger generation of mathematicians throughout the 1950s and 1960s, and was focal in the developments of the preconditions that led to the unprecedented flourishing of mathematical research in Portugal from the 1980s onwards.

The 100th anniversary of Sebastião e Silva's birth is being commemorated with an evocative exhibition at the University of Lisbon. Most of his writings (in particular his research papers mentioned below), which were published at the beginning of the 1980s as a three volume collected works, are now digitally available at

www.sebastiaoesilva100anos.org.

Sebastião e Silva's research

Sebastião e Silva's main research area was functional analysis but his first papers, in 1940, 1941 and 1946, involved a numerical method for the zeros of algebraic equations. Although these were the first research efforts of a young mathematician, their quality and importance is attested by the fact that they were still central in the area of numerical factorisation of polynomials 30 years later [3].

In these early years, Sebastião e Silva also devoted attention to point set topology, then a very active field for the members of the *Movimento Matemático*.

After moving to Italy, Sebastião e Silva started two lines of work: one in mathematical logic and another in functional analysis.

In functional analysis, Sebastião e Silva worked with the Italian mathematician Luigi Fantappiè in the theory of analytic functionals. Sebastião e Silva improved Fantappiè's definition of an analytic functional and, with his new definition, was able to introduce a linear structure on the set of analytic functionals. Next, he introduced the notion of a convergent sequence of analytic functionals. In order to define a topology in the linear space of analytical functionals, he was led to the study of topological vector spaces and, in particular, locally convex spaces. At the same time, he was interested in Laurent Schwartz's theory of distributions.

In the theory of locally convex spaces, Sebastião e Silva studied special cases of inductive and projective limits, defining the spaces LN* and M*, which are known today as Silva spaces.

In the theory of distributions, Sebastião e Silva introduced an axiomatic construction of finite order distributions. This construction was suggested by a previous model of finite order distributions: they are entities of the form D^nF, where F is a continuous function and n is a multi-index. The finite order space of distributions is the quotient space of the Cartesian product of the sets of continuous functions and of non-negative integers by a suitable equivalence relation.

Sebastião e Silva was very proud of his axiomatic construction of distributions. We would like to quote one of his remarks:

> *"Dans le cas de la théorie des distributions, comme dans d'autres cas, les modèles se sont présentés avant l'axiomatique. Et c'est justement la pluralité de concepts concrets, ontologiques, de distribution (comme fonctionnelles, comme séries formelles, comme classes de suites de fonctions, comme couples de fonctions analytiques, etc.), qui suggère d'en extraire la forme abstraite, par axiomatisation. Une définition en plus? Oui et non: il s'agit alors de faire une synthèse des définitions "concrètes"; ce qui en résulte sera plutôt la*

vraie définition." [4] (In the case of distribution theory, as in other cases, the models came before the axiomatic construction. And it is precisely the plurality of concrete, ontological, concepts of distribution (as functionals, as formal series, as classes of sequences of functions, as pairs of analytic functions, etc.), that suggests extracting the abstract form by axiomatization. One more definition? Yes and no: it is rather a synthesis of "concrete" definitions; the result will be the true definition.)

Still in the theory of distributions, Sebastião e Silva introduced the concepts of limit of a distribution at a point, of order of growth of a distribution and of integral of a distribution. The notion of integral allowed him to write the convolution of distributions by the usual formula for functions; a similar situation occurs with the Fourier transform of distributions.

In some problems of differential equations, there is the need for complex translations; they do not exist in the theory of distributions. Also in quantum mechanics there is the need for multipole series; these series are not convergent (unless they reduce to a finite sum) in the theory of distributions. These problems led Sebastião e Silva to the study of ultradistributions. In a paper of 1958 [5], he introduced, using the Stieltjes transform, two new spaces: the space of tempered ultradistributions and the space of exponential growth ultradistributions. The first one contains the space of tempered distributions and the second one the space of distributions of exponential growth. In these spaces, besides the operators of derivation and of product by polynomials, Sebastião e Silva defined the complex translation operator and proved the convergence of some multipole series.

Sebastião e Silva also introduced the space of ultradistributions of compact support. It is a subspace of the space of tempered ultradistributions. In this new space, he conjectured a necessary and sufficient condition for a multipole series to be convergent, the proof of which was completed later by one of his pupils Silva Oliveira.

Sebastião e Silva generalised the Fourier transform to the space of ultradistributions of exponential growth. In that space, this new Fourier transform is a linear and topological endomorphism. This is a beautiful generalisation of Schwartz's result for tempered distributions.

The last research paper written by Sebastião e Silva, when he was hospitalised with a terminal illness, was published posthumously. It deals with an application of tempered distributions with values in a Hilbert space to the Boltzmann equation [6].

Renewal of secondary school instruction in Portugal

Sebastião e Silva's ideas and criticisms about the mathematics being taught in Portuguese schools first appeared in print in the 1940s, in a series of articles published in the *Gazeta de Matemática*. In the 1950s, he wrote two high school textbooks, on algebra and plane analytic geometry, which were later selected by the Ministry of Education for adoption in every school in the country. In this decade, also under his inspiration, Portugal joined the international forums where the renewal of mathematics teaching was being debated. Sebastião e Silva thus represented Portugal in the ICMI (International Commission on Mathematical Instruction) and the CIEAEM (International Commission for the Study and Improvement of Mathematics Teaching).

From 1963 onwards, Sebastião e Silva oversaw a national commission for the modernisation of mathematics teaching in Portuguese secondary schools, which started by dealing with what are nowadays known as the 10th to 12th grades (that is, the three years prior to university entrance). This commission operated under an agreement between the National Ministry of Education and the European Organization for Economic Cooperation (OECE, nowadays OECD). The purpose was to construct a project of reform, part of an international proposal that was considered necessary and urgent to keep up with the evolution of science and technology. The teaching of mathematics was supposed to mirror that evolution, which implied restructuring all the curricula. But, more than that, in the words of Sebastião e Silva [7]:

> *The modernization of the teaching of Mathematics must be done not only regarding curricula but also with respect to teaching methods. The teacher must abandon, inasmuch as possible, the traditional expository method, in which the students' role is almost one hundred percent passive, and try, in contrast, to follow an active method, establishing a dialogue with the students and stimulating their imagination, in such a way as to lead them, whenever possible, to rediscovery.*[1]

The experience of Modern Mathematics, the name by which this project became known, started in the school year 1963/64 in three class groups, taught in three schools (in Lisbon, Porto and Coimbra) by senior teachers who supervised teacher training as part of the commission's work. In the following years, the project was progressively broadened to other schools; new teachers were trained and texts and support materials were written both for students and for teachers of the experimental classes. These texts, *Compêndio de Matemática*[2] and *Guia para a Utilização do Compêndio de Matemática*[3] [7], were written by Sebastião e Silva and are still reference works for the studying and teaching of mathematics at pre-university level.

Lasting influence in Portuguese mathematics

Sebastião e Silva was the foremost Portuguese mathematician in the 20th century and also, undoubtedly, the most influential one. As a professor he was a strong leader of mathematics reform, having a pivotal role in the modernisation of mathematics teaching at all levels. As a researcher, he was at the forefront of several fields in analysis and had a brilliant international career, in particular as the author of an alternative route to developing the theory of distributions. His standing as a researcher may be assessed from the number of items he authored: 49 items listed in ZentralBlatt and 45 in MathSciNet, plac-

ing him as the most productive Portuguese mathematician up to his time.

Most importantly, he was the first Portuguese mathematician of international stature who developed a school of thought, in the sense of leaving a set of disciples capable of carrying on his work and developing his ideas [1]. This is not immediately apparent from the number of PhD students (2) and descendants (10) found in Mathematics Genealogy [8]. There is, however, a reason for this underestimation. Until the late 1960s, a PhD was not mandatory for university professorships in Portugal and so several of his disciples did not earn the degree – but they, in turn, supervised many PhDs. In reality, the number of Sebastião e Silva's direct mathematical descendants is much larger than 10, since, by the next generation, the PhD was already necessary for an academic career.

The impact of the School of Functional Analysis is still felt strongly today, having spread to several Portuguese research universities beyond his original Universidade de Lisboa and Universidade Técnica de Lisboa (which, incidentally, fused in 2013 to form the new ULisboa). In fact, it may be argued, without great exaggeration, that the origins of functional analysis in Portugal may be traced back to a single point: Sebastião e Silva.

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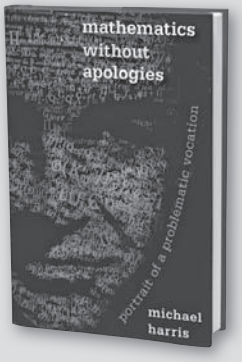
[2] Compendium of Mathematics.
[3] Guide for the Use of the Compendium of Mathematics.

Jorge Buescu [jsbuescu@fc.ul.pt] is a professor of mathematics at the Faculty of Science of the University of Lisbon. Besides his mathematical activity on dynamical systems and analysis, he is highly engaged in the popularisation of mathematics, being the author of several books. He is Vice-President of the Portuguese Mathematical Society, a member of the EMS RPA Committee and an Editor of the EMS Newsletter.

Luís Canto de Loura [cantoloura@gmail.com] is retired from the Instituto Superior de Engenharia de Lisboa, where he was a full professor and President of the Departamento de Matemática (2006–2011). From 1993 to 2006 he was President of the Departamento de Matemática of the Faculdade de Motricidade Humana. From 1971 to 1993 he was a member of the Departamento de Matemática of the Instituto Superior Técnico. He holds a PhD in numerical analysis from Houston University and a degree of Docteur de troisième cycle from the Université de Paris VI.

Fernando P. da Costa [fcosta@uab.pt] teaches at the Department of Sciences and Technology of Universidade Aberta, Lisbon, and is a researcher at the CAMGSD, IST, University of Lisbon, Portugal. His research areas are analysis and differential equations. He got his PhD at Heriot-Watt University, Edinburgh, in 1993, and he is the current President of the Portuguese Mathematical Society.

Anabela A. Teixeira [ateixeira@museus.ul.pt] is a teacher of mathematics in secondary education, assigned to Associação Ludus; she also works at the Museu Nacional de História Natural e da Ciência da Universidade de Lisboa. She holds an MSc in Mathematics from the University of Coimbra and is finishing her PhD on History of Education at the University of Lisbon.



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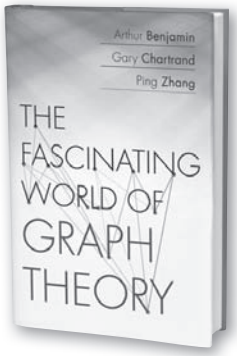

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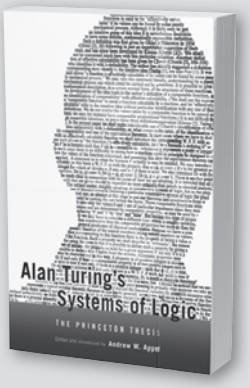

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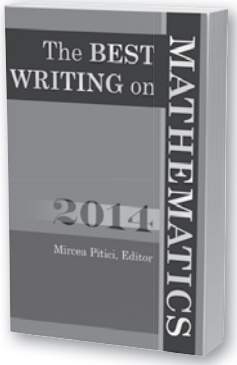

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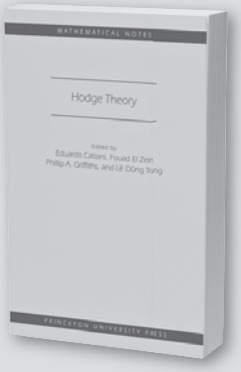

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Grothendieck and Algebraic Geometry

Luc Illusie and Michel Raynaud[1] (both Université Paris-Sud, Orsay, France)

Grothendieck's thesis and subsequent publications in the early 1950's dealt with functional analysis. This was remarkable work, which is attracting new attention today[2]. Still, his most important contributions are in algebraic geometry, a field which occupied him entirely from the late 1950's on, in particular during the whole time he was a professor at the IHÉS (1959-1970).

Algebraic geometry studies objects defined by polynomial equations[3] and interprets them in a geometric language. A major problem faced by algebraic geometers was to define a good framework and develop local to global techniques. In the early 1950's complex analytic geometry showed the way with the use of sheaf theory. Thus, a complex analytic space is a ringed space, with its underlying space and sheaf of holomorphic functions (Oka, H. Cartan). Coherent sheaves of modules over this sheaf of rings play an important role[4]. In 1954 Serre transposed this viewpoint to algebraic geometry for varieties defined over an algebraically closed field. He employed the Zariski topology, a topology with few open subsets, whose definition is entirely algebraic (with no topology on the base field), but which is well adapted, for example, to the description of a projective space as a union of affine spaces, and gives rise to a cohomology theory which enabled him, for example, to compare certain algebraic and analytic invariants of complex projective varieties.

Inspired by this, Grothendieck introduced *schemes* as ringed spaces obtained by gluing (for the Zariski topology) spectra of general commutative rings. Furthermore, he described these objects from a *functorial* viewpoint. The language of categories already existed, having appeared in the framework of homological algebra, following the publication of Cartan-Eilenberg's book (*Homological Algebra*, Princeton Univ. Press, 1956). But it was Grothendieck who showed all its wealth and flexibility. Starting with a category C, to each object X of C one can associate a *contravariant functor* on C with values in the category of sets, $h_X : C \to Sets$, sending the object T to $\mathrm{Hom}_C(T, X)$. By a classical lemma of Yoneda, the functor $X \mapsto h_X$ is fully faithful. To preserve the geometric language, Grothendieck called $h_X(T)$ the set of *points* of X with values in T. Thus, an object X is known when we know its points with values in every object T. Grothendieck applied this to algebraic geometry. This was revolutionary as, until then, only field valued points had been considered.

As an example, suppose we have a system of equations

$$f_1(x_1, \cdots, x_n) = \cdots = f_N(x_1, \cdots, x_n) = 0, \tag{1}$$

where the f_i's are polynomials with coefficients in $\mathbf{Z}$. Let $\mathcal{A}$ be the opposite category of the category of rings, and let F be the (contravariant) functor sending a ring A to the set $F(A)$ of solutions $(x_i), x_i \in A$, of (1). This functor is nothing but the functor h_X for X the object of $\mathcal{A}$ corresponding to the quotient of $\mathbf{Z}[x_1, \cdots, x_n]$ by the ideal generated by the f_i's: the functor F is *represented* by X, an *affine scheme*. Points of X with values in $\mathbf{C}$ are points of a complex algebraic variety — that one can possibly study by analytic methods — while points with values in $\mathbf{Z}$, $\mathbf{Q}$, or in a finite field are solutions of a diophantine problem. Thus the functor F relates arithmetic and geometry.

If the f_i's have coefficients in a ring B instead of $\mathbf{Z}$, the analogous functor F on the category $\mathcal{A}$ opposite to that of B-algebras, sending a B-algebra A to the set $F(A)$ of solutions of (1) with values in A, is similarly represented by the spectrum X of a B-algebra (quotient of $B[x_1, \cdots, x_n]$ by the ideal generated by the f_i's), a scheme *over* the spectrum of B. In this way a *relative* viewpoint appears, for which the language of schemes is perfectly suited. The essential tool is *base change*, a generalization of the notion of extension of scalars: given a scheme X over S and a base change morphism $S' \to S$, we get a new scheme X' over S', namely, the fiber product of X and S' over S. In particular, X defines a family of schemes X_s parametrized by the points s of S. The above functor F then becomes the functor sending an S-scheme T to the set of S-morphisms from T to X. A number of useful properties of X/S (such as smoothness or properness) can nicely be read on the functor h_X. The two above mentioned properties are stable under base change, as is *flatness*, a property that plays a central role in algebraic geometry, as Grothendieck showed. In 1968, thanks to M. Artin's approximation theorem, it became possible to characterize functors that are representable by algebraic spaces (objects very close to schemes) by a list of properties of the functor, each of them often being relatively easy to check. But already in 1960, using only the notion of flatness, Grothendieck had constructed, in a very natural way, Hilbert and Picard schemes as representing certain functors, at once superseding – by far – all that had been written on the subject before.

Nilpotent elements[5] in the local rings of schemes appear naturally (for example in fiber products), and they play a key

1 We thank Jean-Benoît Bost, Pierre Deligne, and Jean-Pierre Serre for their remarks on a preliminary French version of this text. The first author thanks Robin Hartshorne, Nicholas Katz, William Messing, and Arthur Ogus for their help with its translation into English.

2 See [G. Pisier, *Grothendieck's theorem, past and present*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 2, 237–323].

3 However, we seldom saw Grothendieck write an explicit equation on the blackboard; he did it only for basic, crucial cases.

4 Cf. Cartan's famous theorems A and B.

5 An element x of a ring is called *nilpotent* if there exists an integer $n \geq 1$ such that $x^n = 0$.

role in questions of infinitesimal deformations. Using them systematically, Grothendieck constructed a very general differential calculus on schemes, encompassing arithmetic and geometry.

In 1949 Weil formulated his conjectures on varieties over finite fields. They suggested that it would be desirable to have at one's disposal a cohomology with discrete coefficients satisfying an analogue of the Lefschetz fixed point formula. In classical algebraic topology, cohomology with discrete coefficients, such as $\mathbf{Z}$, is reached by cutting a complicated object into elementary pieces, such as simplices, and studying how they overlap. In algebraic geometry, the Zariski topology is too coarse to allow such a process. To bypass this obstacle, Grothendieck created a conceptual revolution in topology by presenting new notions of gluing (a general theory of *descent*[6], conceived already in 1959), giving rise to new spaces: *sites* and *topoi*, defined by what we now call *Grothendieck topologies*. A Grothendieck topology on a category is the datum of a particular class of morphisms and families of morphisms $(U_i \to U)_{i\in I}$, called *covering*, satisfying a small number of properties, similar to those satisfied by open coverings in topological spaces. The conceptual jump is that the arrows $U_i \to U$ are not necessarily inclusions[7]. Grothendieck developed the corresponding notions of sheaf and cohomology. The basic example is the *étale topology*[8]. A seminar run by M. Artin at Harvard in the spring of 1962 started its systematic study. Given a scheme X, the category to be considered is that of étale maps $U \to X$, and covering families are families $(U_i \to U)$ such that U is the union of the images of the $U_i's$. The definition of an étale morphism of schemes is purely algebraic, but one should keep in mind that if X is a complex algebraic variety, a morphism $Y \to X$ is étale if and only if the morphism $Y^{\mathrm{an}} \to X^{\mathrm{an}}$ between the associated analytic spaces is a local isomorphism. A finite Galois extension is another typical example of an étale morphism.

For torsion coefficients, such as $\mathbf{Z}/n\mathbf{Z}$, one obtains a good cohomology theory $H^i(X, \mathbf{Z}/n\mathbf{Z})$, at least for n prime to the residue characteristics of the local rings of X. Taking integers n of the form ℓ^r for a fixed prime number ℓ, and passing to the limit, one obtains cohomologies with values in $\mathbf{Z}_\ell = \varprojlim \mathbf{Z}/\ell^r\mathbf{Z}$, and its fraction field $\mathbf{Q}_\ell$. If X is a complex algebraic variety, one has comparison isomorphisms (due to M. Artin) between the étale cohomology groups $H^i(X, \mathbf{Z}/\ell^r\mathbf{Z})$ and the Betti cohomology groups $H^i(X^{\mathrm{an}}, \mathbf{Z}/\ell^r\mathbf{Z})$[9], thus providing a purely algebraic interpretation of the latter. Now, if X is an algebraic variety over an arbitrary field k (but of characteristic $\neq \ell$) (a k-scheme of finite type in Grothendieck's language), $\bar{k}$ an algebraic closure of k, and $X_{\bar{k}}$ deduced from X by extension of scalars, the groups $H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$ are finite dimensional $\mathbf{Q}_\ell$-vector spaces, and they are equipped with a continuous action of the Galois group $\mathrm{Gal}(\bar{k}/k)$. It is especially through these representations that algebraic geometry interests arithmeticians. When k is a finite field $\mathbf{F}_q$, in which case $\mathrm{Gal}(\bar{k}/k)$ is generated by the Frobenius substitution $a \mapsto a^q$, the Weil conjectures, which are now proven, give a lot of information about these representations. Étale cohomology enabled Grothendieck to prove the first three of these conjectures in 1966[10]. The last and most difficult one (*the Riemann hypothesis for varieties over finite fields*) was established by Deligne in 1973.

When Grothendieck and his collaborators (Artin, Verdier) began to study étale cohomology, the case of curves and constant coefficients was known: the interesting group is H^1, which is essentially controlled by the jacobian of the curve. It was a different story in higher dimension, already for a surface, and *a priori* it was unclear how to attack, for example, the question of the finiteness of these cohomology groups (for a variety over an algebraically closed field). But Grothendieck showed that an apparently much more difficult problem, namely a relative variant of the question, for a morphism $f : X \to Y$, could be solved simply, by *dévissage* and reduction to the case of a family of curves[11]. This method, which had already made Grothendieck famous with his proof, in 1957, of the *Grothendieck-Riemann-Roch formula* (although the dévissage, in this case, was of a different nature), suggested a new way of thinking, and inspired generations of geometers.

In 1967 Grothendieck defined and studied a more sophisticated, second type of topology, the *crystalline topology*, whose corresponding cohomology theory generalizes de Rham cohomology, enabling one to analyze differential properties of varieties over fields of characteristic $p > 0$ or p-adic fields. The foundations were written up by Berthelot in his thesis. Work of Serre, Tate, and Grothendieck on p-divisible groups, and problems concerning their relations with Dieudonné theory and crystalline cohomology launched a whole new line of research, which remains very active today. *Comparison theorems* (solving conjectures made by Fontaine[12]) establish bridges between étale cohomology with values in $\mathbf{Q}_p$ of varieties over p-adic fields (with the Galois action) on the one hand, and their de Rham cohomology (with certain extra structures) on the other hand, thus providing a good understanding of these p-adic representations. However, over global fields, such as number fields, the expected properties of étale cohomology, hence of the associated Galois representations, are still largely conjectural. In this field, the progress made since 1970 owes much to the theory of automorphic forms (*the Langlands program*), a field that Grothendieck never considered.

In the mid 1960's Grothendieck dreamed of a universal cohomology for algebraic varieties, without particular coefficients, having realizations, by appropriate functors, in the cohomologies mentioned above: the theory of *motives*. He gave a construction, from algebraic varieties and algebraic correspondences between them, relying on a number of conjec-

6 The word "descent" had been introduced by Weil in the case of Galois extensions.

7 more precisely, monomorphisms, in categorical language.

8 The choice of the word *étale* is due to Grothendieck.

9 but not between $H^i(X, \mathbf{Z})$ and $H^i(X^{\mathrm{an}}, \mathbf{Z})$: by passing to the limit one gets an isomorphism between $H^i(X, \mathbf{Z}_\ell)$ and $H^i(X^{\mathrm{an}}, \mathbf{Z}) \otimes \mathbf{Z}_\ell$.

10 The first one (*rationality of the zeta function*) had already been proved by Dwork in 1960, by methods of p-adic analysis.

11 at least for the similar problem concerning *cohomology with proper supports*: the case of cohomology with arbitrary supports was treated only later by Deligne using other dévissages.

12 the so-called C_{cris}, C_{st}, and C_{dR} conjectures, first proved in full generality by Tsuji in 1997, and to which many authors contributed.

tures that he called *standard*. Except for one of them[13], they are still open. Nevertheless, the dream was a fruitful source of inspiration, as can be seen from Deligne's theory of absolute Hodge cycles, and the construction by Voevodsky of a triangulated category of mixed motives. This construction enabled him to prove a conjecture of Bloch-Kato on Milnor K-groups, and paved the way to the proof, by Brown, of the Deligne-Hoffman conjecture on values of multizeta functions.

The above is far from giving a full account of Grothendieck's contributions to algebraic geometry. We did not discuss Riemann–Roch and K-theory groups, stacks and gerbes[14], group schemes (SGA 3), derived categories and the formalism of six operations[15], the *tannakian* viewpoint, unifying Galois groups and Poincaré groups, or *anabelian geometry*, which he developed in the late 1970's.

All major advances in arithmetic geometry during the past forty years (proof of the Riemann hypothesis over finite fields (Deligne), of the Mordell conjecture (Faltings), of the Shimura–Taniyama–Weil conjecture (Taylor-Wiles), works of Drinfeld, L. Lafforgue, Ngô) rely on the foundations constructed by Grothendieck in the 1960's. He was a visionary and a builder. He thought that mathematics, properly understood, should arise from "natural" constructions. He gave many examples where obstacles disappeared, as if by magic, because of his introduction of the right concept at the right place. If during the last decades of his life he chose to live in extreme isolation, we must remember that, on the contrary, between 1957 and 1970, he devoted enormous energy to explaining and popularizing, quite successfully, his point of view.

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Luc Illusie and Michel Raynaud are both honorary professors at Université Paris-Sud. They were both PhD students of Alexander Grothendieck and they have made notable contributions to algebraic geometry.

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13 the *hard Lefschetz* conjecture, proved by Deligne in 1974.

14 These objects had been introduced by Grothendieck to provide an adequate framework for non abelian cohomology, developed by J. Giraud (*Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften **179**, Springer-Verlag, 1971). Endowed with suitable algebraic structures (Deligne-Mumford, Artin), stacks have become efficient tools in a lot of problems in geometry and representation theory.

15 Currently used today in the theory of linear partial differential equations.

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Interview with Fields Medallist Artur Avila

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden)

My standard question: were you surprised that you got the Fields Medal and how will it affect your life?
I certainly did not expect to get the medal. In fact, I thought it was more likely to happen back in Hyderabad. Now, I knew that Maryam Mirzakhani was considered and I thought that, in this case, I was more unlikely to be awarded.

You mean they would not award it to two people doing billiards?
We do much more than billiards. As to changing my life, I hope it will not get in my way as to doing research. But, of course, with the ICM in Rio, my celebrity status is bound to be exploited.

How did it feel sitting there on the podium?
A bit stressful. We had rehearsed the whole thing the evening before and I was very much concentrating on doing the right thing.

Yes, it has become more and more of a show. And the media attention seems to have increased dramatically, even since Hyderabad. I do not know whether this is due to it being in Korea. There are rumours that you got a thousand emails a day.
I did not count them but I surely got swamped; there is no way I can systematically go through them.

When did you get interested in mathematics?
I have been interested in mathematics, numbers and such, as long as I can remember. All my life essentially, I guess.

What about your background? What are your parents doing?
They are actually retired now but they used to work as bureaucrats. There was no connection at all to academia.

And when did you get seriously interested (I mean considering becoming a mathematician)?
When I was 13, I was asked by my teacher to participate in the mathematical Olympiad.

This was a local Olympiad? The Brazilian one?
That is true. I did it at first without any preparation. Then, next year I tried again and I got involved in the Olympiad programme, when I was 15 or so. This was also the time I became aware of IMPA.

How do you look upon mathematical Olympics and, in particular, the competitive aspects of mathematics? Is that important?

Photo: Alfredo Brant

I think the main point about the mathematical Olympiads is that it makes you encounter interesting problems, problems you would not encounter at school. The competitive aspect is secondary but, of course, competition comes naturally when you are young. After all, young people compete in sports all the time. But I would say that after a certain stage in your mathematical career, it becomes irrelevant. As a mathematician, you are free to do your own thing and usually there are very few people who are really interested in what you are doing, and you would rather cooperate with them instead.

You completed your PhD at 21. Was it an advantage being so young? And how come you did it so quickly?
Having become aware of IMPA, I knew that they had a pre-Master's programme to which I applied, which naturally led to a PhD.

People have wondered why you did not go to some prestigious place in Europe for your PhD.
I did not plan my career at all. One thing led to another. It just seemed very natural for me to continue at IMPA. In retrospect, I can see that it was probably a very good choice at the time. It provided a stimulating atmosphere without any pressure.

You said that your natural taste is for analysis, not algebra. What is it about algebra that you dislike?
I do not dislike it. In fact, when I studied I was quite good at it but I always felt that it was sort of mechanical. I learned it. It did not come from the inside. I lacked intuition. It was different with analysis. I naturally decided that if I did something, it better be with something for which I had a natural affinity.

What is the nature of your intuition? Is it geometrical?
It is definitely geometrical. I need to visualise but, of course, when it comes down to actually doing it, you need to make various estimates.

You got into dynamical systems very early in your career. How did that happen?
It was because of Welington de Melo. I started to talk to him and he was very helpful and we got along and, once again, without actually planning anything, I ended up being his student.

While you were doing your PhD, or even before, were you very ambitious? Did you dream about getting the Fields Medal?
I would not say I was really ambitious. My immediate goal was to finish my thesis and to get a job. As to Fields Medals, I certainly knew of them as Yoccoz had visited IMPA and, when McMullen got his in 1998, I was told 'there is someone in your field actually getting a medal'.

And then you went to Paris. Would it be fair to say that this was when your real education began?
My mathematical education certainly did not stop with the completion of my PhD. In Paris, my views certainly enlarged and I got a much better perspective, and certainly I learned a lot. But even if my general mathematical education might have been limited, I knew that I knew some things very deeply, and actually having solved a problem and written a thesis gave me a lot of confidence.

I think this is very important. Many ambitious people may try to get a big overview of mathematics before they start to do research, and this can have a very inhibiting effect. Your own efforts would seem so puny in comparison with what has already been done and you start worrying about not being able to write a thesis at all, which can develop into a neurosis.
As I pointed out, we all worry about not being able to write a thesis. It is normal. As to it becoming a neurosis, I have no experience of that.

So you always had a very strong confidence in your abilities? You were never overwhelmed by frustrations and never considered dropping out of mathematics?
There are always frustrations in mathematics. But those are always local, at least in my personal experience.

So what did you do arriving in Paris?
I actually enrolled in a study group organised by Krikorian and Eliasson. Those are fairly common in Paris and anyone can participate. It was very helpful. I followed the presentations, asking questions.

How do you actually learn mathematics?
Not by reading. At least, I never read any books. Articles I sort of read, meaning that I quickly skim through them looking for crucial points where things 'are happening' so to speak. Basically, I learn all of my mathematics from talking to people.

So personal conversations are the most efficient way of conveying mathematical insights?
They definitely are.

So why is that?
They are informal for one thing. I always find it hard to express what I am thinking. In a conversation you can do it by appealing to a common understanding. And you do not have to be systematic; you can concentrate on the crucial bits I referred to above.

You spoke earlier of your inability to read mathematics. What about writing mathematics?
It depends on the subject. Some of my papers write themselves very easily but when it comes to classical dynamical problems in one real or complex variable, I find that those are very hard, and I have a very hard time expressing myself. It is very difficult to convey my geometric intuition.

Have you ever had any competing interests?
When I was very young, I was interested in science in general but, of course, at the age of 13 or so those interests are very superficial and soon my interest was focused entirely on mathematics.

But you cannot be doing mathematics all the time. What are you doing when you are not doing mathematics?
As that video at the opening ceremony showed, I like to go to the beach, being a Brazilian boy, and I like to go to the gym. Of course, sometimes I do mathematics on the beach.

…like Feynman and Steven Smale. Sorry – please go on…
But I never think of mathematics when I am working out in the gym. Then, of course, it also depends on where I am. I do slightly different things back in Brazil from what I do in Paris. I like going to bars and meeting friends, sometimes talking mathematics but normally not.

They did not show that in the video. So what are you normally talking about when you hang out in bars? Women and soccer?
I am not interested in soccer.

And you are Brazilian?
Most of my friends are mathematicians so mathematics is a natural topic.

Do you run?
Run? No. Not at all. It does not appeal to me. In the gym, I am into weightlifting but I do not take it as seriously as some of my friends, who are forever discussing how much they lift and what diets to keep to.

What about other interests? Do you read? Do you like music?
I do not read at all. As to music, I have my tastes and in the past I used to go to the opera and such things. Not

anymore. I feel the need more and more to meet friends and to hang out with them. This kind of social life is very important to me.

You spend about half of your time in Brazil and half in Paris. Does that affect you and is your life different depending on where you are?
It is an arrangement I am very happy with and which I have no desire to change. I spend about six months at either place. Of course my life is different depending on where I live. In Brazil, I am much more relaxed – for one thing I do not have to commute. I do not like to spend time in transportation as I am forced to do in Paris. And, of course, life in Paris is more pressured in other ways too.

What is your stand on pure mathematics versus applied mathematics? People never ask about the applications of mathematics to soccer or music. Why should mathematicians have to justify themselves?
I am a pure mathematician and I do not feel the need to justify this. Of course, any piece of mathematics may be applicable but such things are unpredictable, so there really is no alternative to mathematicians following their intrinsic interests. Furthermore, mathematics is cheap; it is the science that requires the least resources, so in that sense it does make perfect sense for poorer Third World countries to invest in mathematics. It also breeds a general culture conducive to science as a whole. Thus, it is great that Rio will host the congress next time around.

Have you ever been involved in mathematical education and trying to reach larger audiences?
I have been lucky for most of my career; I have had to do no basic teaching, just advising doctoral students and typically lecturing on my own work. I did, of course, some basic teaching when I started out at IMPA. At 18, I was grading exercises in linear algebra and one of the students was just 13. He ended up getting a PhD at 19 and is now employed by the CNRS.

Hardly a typical teaching experience.

Ulf Persson [ulfp@chalmers.se] has been a member of the editorial board of the EMS Newsletter since 2006 and a professor of mathematics at Chalmers University of Technology in Gothenburg, Sweden, since 1989, receiving his Ph.D. at Harvard in 1975. He has in the past interviewed recent Fields medalists specifically for the EMS Newsletter, as well as other mathematicians for alternate assignments, in a conversational style, some published, others as of yet unpublished. There is a plan to make a selection and collect them into a forthcoming book.

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Solid Findings: Students' Over-reliance on Linearity[1]

Lieven Verschaffel (KU Leuven, Belgium) on behalf of the Education Committee of the EMS

Introduction

One of the major goals of elementary and middle grade mathematics education is for students to gain a deep understanding of the linear model in a variety of forms and applications. By the term *linear*[2] we refer to functions of the form $f(x) = ax$. There are several well-known and frequently used aspects, properties and representations of such functions that are emphasised at many points in mathematics curricula. A thorough insight into each of these and the links between them is considered to be a deep understanding of linearity. The first aspect is about the graphical representation, which is a straight line through the origin. As etymology indicates, linearity (derived from the Latin adjective "linearis", from "linea", line) refers to the idea of a line, that is, in usual parlance, a straight line.[3,4] However, despite common usage, it is equally important to understand that not every function whose graph is a straight line is a linear function in our sense. Secondly, linear functions are closely related to the idea of proportion ($a/b = c/d$), which is the basis of a wide variety of mathematics problems, such as the following missing-value problem: "Yesterday I made lemonade using 5 lemons in 2 litres of water. How many lemons do I need in 1 litre of water if I want the same taste?"). Thirdly, linear functions have properties such as $f(kx) = kf(x)$, which is often presented in the classroom when dealing with relatively simple missing-value problems and is then worded as a "k times A, k times B" rule (for instance "to cover 3 times as much area, I need 3 times as much paint"), or the additive property that $f(x+y) = f(x) + f(y)$, which may appear as a "Theorem-in-action" (Vergnaud, 1983) in many cases, for instance when children use a "building-up" approach to compute the answer of a missing-value problem (e.g. "to cover $6+6+6$ m², I need $0.75+0.75+0.75$ litres of paint").

Over-reliance on linearity – some examples from the research literature

As shown below, research points out that there is a strong and resistant tendency among many students – and even in adults – to see and apply the linear properties described above everywhere, and that this tendency to over-use linearity becomes more pronounced with students' mastery of the linearity concept and the accompanying computational techniques. The first curricular domain where this tendency clearly manifests is arithmetic word problems. When confronted with problems that do not allow for a simple, straightforward numerical answer due to certain, content-specific, realistic constraints, such as "John's best time to run 100 metres is 17 seconds; how long will it take him to run 1 kilometre?", nearly all upper elementary school students and most student-teachers answer with the straightforward, computational answer "$17 \times 10 = 170$ seconds" without any concern for the non-realistic nature of their reaction (Verschaffel et al., 2000). But, even more strikingly, students also give unwarranted proportional answers to arithmetic word problems for which there is clearly a correct, non-proportional answer. Cramer et al. (1993), for instance, confronted pre-service elementary teachers with the additive problem: "Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?" (p. 160). Nearly all pre-service teachers solved this problem by setting up and solving a proportion: $9/3 = x/15$; $3x = 135$; $x = 45$, instead of noticing that Sue has always run 6 laps more than Julie.

A second domain in which students massively fall back on simple linear methods involves the relationship between the lengths and the area or volume of geometrically similar figures. In a series of studies, De Bock et al. (2007) administered paper-and-pencil tests with proportional as well as non-proportional word problems about the lengths, perimeters, areas and volumes of different types of figures to large groups of 8th- to 10th-graders. An example of a non-proportional problem is: "Farmer Carl needs approximately 8 hours to fertilise a square pasture with a side of 200 m. How many hours would he need to fertilise a square pasture with a side of 600 m?" More than 90% of the 8th-graders and more than 80% of the 10th-graders failed on this type of non-proportional problem because of their tendency to apply proportion-

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1 This "solid finding" paper is largely based on the book of De Bock et al. (2007).

2 Generally the name "linear" is given to functions of the form $f(x) = ax + b$. Such functions are graphically represented by a straight line that does not pass through the origin, except in the special case where $b = 0$. In this case, the function is said to be homogeneous. However, following De Bock et al. (2007), we will mean by a linear function a homogeneous linear function, i.e. one of the form $f(x) = ax$.

3 In the special case of $a = 0$, the "geometry" of the linear function is a horizontal line.

4 The use of the term "line" for straight line and "curve" for other lines is fairly recent. Up to the 19th century, "(curved) line" meant what we now mean by "curve". From the Greek up to at least the 17th century (but, to some extent, even until the early 19th century), it was a tradition to divide geometrical problems into three categories: (1) plane problems (using only straight lines and circles), (2) solid problems (using conics) and (3) linear problems (dealing with all other lines, i.e. curves) (see, for example, Rabuel's "Commentaires sur la géométrie de Descartes").

al methods. In later studies, De Bock et al. found that various kinds of help resulted in no decrease or only a marginal decrease in the percentage of proportional solutions to these non-proportional geometry problems.

Thirdly, the domain of probabilistic reasoning has also been shown to be sensitive to the presence of the same phenomena, that is, improper applications of linearity. Consider, for example, the coin problem that Fischbein (1999) gave to 5th- to 11th-graders: "The likelihood of getting heads at least twice when tossing three coins is smaller than / equal to / greater than the likelihood of getting heads at least 200 times out of 300 times" (p. 45). Actually, it is very likely to get two or three heads when tossing three coins but insight into the laws of large numbers would reveal that getting 200 or more heads over 300 coin tosses is very unlikely. Nevertheless, Fischbein found that the number of erroneous answers of the type "equal to" increased with age: 30% in Grade 5, 45% in Grade 7, 60% in Grade 9 and 80% in Grade 11. Van Dooren et al. (2003) also found that a large number of erroneous reasonings in the domain of probability can be explained by students' assuming a linear relation between the variables determining a binomial chance situation.

Finally, in the domain of number patterns, algebra and calculus, Küchemann and Hoyles (2009) gave high-achieving 8th- to 10th-graders in England an example of a relationship showing a series of 6 grey tiles surrounded by a single layer of 18 white tiles and asked them to generalise this to another number (60) of white tiles. Altogether, 35% of students gave erroneous linear responses in Grade 8 and this percentage only slightly decreased to 21% in Grade 10. It is also not unusual to see students believing that transcendental functions are linear, like $sin(ax) = a\ sin\ x$ or $log\ (ax) = a\ log\ x$ (De Bock et al., 2007).

In search of explanations

The roots of the over-use of linearity lie, first of all, in the intuitive character of human cognition in general and of mathematical reasoning in particular (Fischbein, 1999; Kahneman, 2002). Mathematically, a function of the form $f(x)=ax$ is one of the simplest relationships that can occur between two variables. And, from a developmental perspective, students begin to master problem situations with small integer proportionality factors well before instruction in formal linear reasoning has even started. For Fischbein (1978, 1999), intuitive knowledge is a type of immediate, implicit, self-evident cognition, based on salient problem characteristics, leading in a coercive manner to generalisations, generating great confidence and often persisting despite formal learning. Empirical indications for the intuitiveness of linearity are found in a study of De Bock et al. (2002), who studied 7th- to 10th-grade students' over-reliance on direct proportionality when solving geometry problems through individual interviews. They found not only that students were very quick to give a proportional answer to geometry problems, such as the farmer problem mentioned above, but also that students found it very difficult to justify their choice, were absolutely convinced about its correctness, and could not think of any possible alternative. And afterwards, when the interviewer confronted them with a correct alternative, even with explicit justifications for it, many students remained extremely reluctant to abandon their initial solution.

Another, probably even more important, explanation for students' tendency to over-rely on linearity can be found in the experiences they have had during their mathematics lessons. At certain moments in the mathematics curriculum, extensive attention is paid to linearity, to its properties or representations, and to the fluent execution of certain linearly-based computations, without questioning whether the property, representation or computation is applicable. Moreover, the general classroom culture and practice of not stimulating students to pay attention to the relationship between mathematics and the real world further shapes students' superficial and routine-based problem-solving tactics, wherein there is little room for questioning the kind of mathematical model for the problem situation at hand (Verschaffel et al., 2000).

Before commenting on some educational implications, we point out that students are not the only people that fall into the linearity trap. Politicians, economists and the media also tend to take it for granted that quantitative problems always have solutions rooted in linearity (De Bock et al., 2007). Of course, mathematicians have for a long time used linear models even for the study of complex systems (Poincaré, 1905) and still frequently assume linearity to simplify in order to model. "Going linear" is then a good (first) approach. The important difference is that "mathematicians do so mindfully, explicitly stating what simplifying assumptions are being made, and with a feel for the degree of inaccuracy that the simplification introduces relative to the goals of the exercise" (Verschaffel et al., 2000, p. 167).

Implications for education

Probably the most important educational steps towards challenging students' tendency to over-use linear methods are to introduce more variation in the mathematical tasks given to students and to approach these tasks from a mathematical modelling perspective rather than a computational perspective. More specifically, textbook writers and teachers should, firstly, restrain from only working with problems that allow students to come up with a correct solution while harbouring an incorrect general strategy. Rather, they should confront students regularly with mathematically different problems within the same context and/or presentational structure. Secondly, forms of answer other than exact numerical answers could be used much more frequently, e.g. making estimations, commentaries, drawings, graphs, etc. Alternative forms of tasks such as classification tasks and problem-posing tasks (instead of traditional tasks whereby the student "only" has to solve the problem) could also be included. Such alternative forms of answers and tasks help to move the students' attention from individually calculating numerical answers towards classroom discussions on the link between problem situations and arithmetical op-

erations. Thirdly, increasing the authenticity of the word problems may further help to shift the disposition of the students toward genuine modelling, with the idea of linearity seen as a specific functional dependency as a way to overcome thinking in terms of proportion. This means moving from thinking of proportion as two ordered pairs of numbers in the same ratio to thinking in terms of variables related by a multiplicative factor. The specific two pairs of numbers forming a specific proportion have to become part of an infinite set of ordered pairs of numbers, and all these pairs may be suitably selected to form a proportion. Such a move from proportional reasoning to functional reasoning, relating variables in a linear multiplicative way, could be fostered by developing a modelling approach, rather than a more traditional approach consisting of proposing specific missing-value problems.

However, the available research also suggests that students' over-use of linearity cannot be prevented or remedied just by means of simple, short-term interventions. It needs long-term and systematic attention from an early start. So, already when introducing and practising the basic mathematical operations in the first years of elementary school, mathematics educators have to pay attention to the fact that these operations can model some situations but not others (Verschaffel et al., 2000).

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the European Mathematical Society. The committee members are Franco Brezzi, Qendrim Gashi, Ghislaine Gueudet, Bernard Hodgson, Celia Hoyles, Boris Koichu, Konrad Krainer, Maria Alessandra Mariotti, Mogens Niss, Juha Oikonnen, Nuria Planas, Despina Potari, Alexei Sossinsky, Ewa Swoboda, Günter Törner and Lieven Verschaffel.

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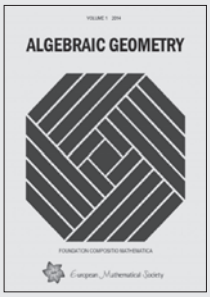

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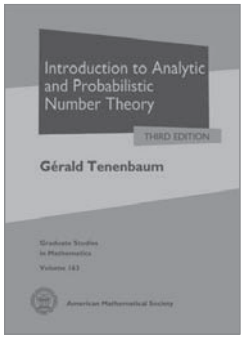

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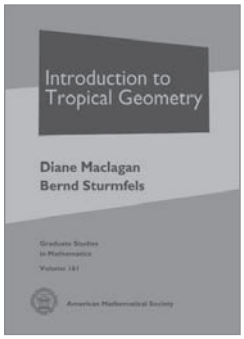

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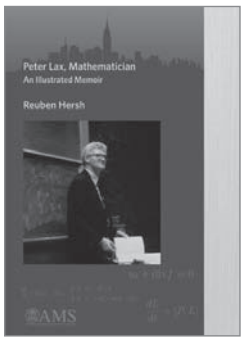

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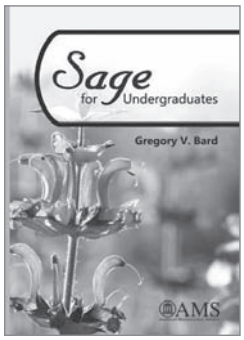

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A Glimpse at the Reviewer Community

Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

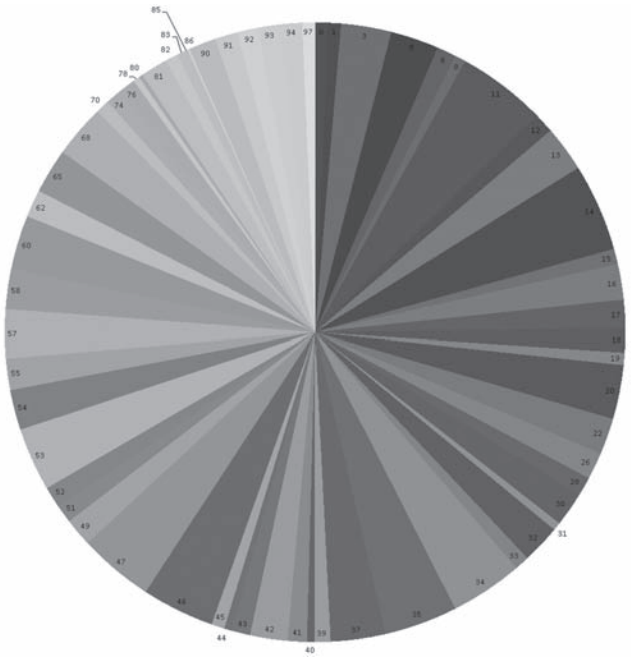
One of the moments which makes the ICM such a special event is the gathering of the global reviewing community (well, at least quite a representative part of it). While occasions like the Joint Mathematics Meeting and the European Congress provide platforms for regional meetings, and are accompanied by regular receptions of MR or zbMATH reviewers, respectively, only the International Congress (every four years) brings people together from the community of more than 20,000 mathematicians who do active reviewing.

An impression from the Joint Reviewer Reception of MathSciNet and zbMATH at the ICM in Seoul

While making up only a fraction of the global mathematical community (which accounts for more than 100,000 scientists, when applications are included), reviewers still account for a significant part. And while there is obviously no archetypical reviewer – the community is almost as diverse as mathematics itself – one is tempted to notice some common ground at such meetings. In the following, we try to give some figures about the zbMATH reviewer community.

Reviewing was established as a form of mathematical dissemination in the 19th century. From the beginning, not only books (though they formed a much larger part of the literature back then) but also articles and conference contributions were reviewed, since it turned out that it was very helpful to have publication content summarised and put into the context of the overall research by independent experts. Though one has to keep in mind that there were usually no abstracts at that time, which now fulfil a part of this task, it has turned out that reviewing is still a viable approach to mathematical communication: it gives independent information, gives broader context, indicates further hints and problems and sometimes even points out possible lack of clarity, gaps or (in extreme cases) lack of novelty or suspected plagiarism. Moreover, the reviewer often comes across new ideas for their own research.

Naturally, the amount of reviewing activity is closely related to the mathematical subject and also somewhat to the longevity of the results – since reviews take their time, they appear less frequently in fast-paced areas where abstracts are often considered to be sufficient. For instance, in algebraic geometry, about 85% of recent publications in zbMATH will find a reviewer, compared to less than 10% in solid and fluid mechanics, control theory or statistics. This is reflected by the structure of the reviewer community: of about 7,200 active zbMATH reviewers (plus several hundred on a temporary pause), most are in number theory and algebraic geometry (both 11%, while these areas make up only about 1% and 3% of the overall publications, respectively), followed by 10% of reviewers in the three areas of PDEs, functional analysis and operator theory (with publication shares of 6%, 1%, and 2%). Obviously, this also affects the quota of reviews, which is about 25% overall but larger than 50% for the "lower" MSC areas (up to 60).

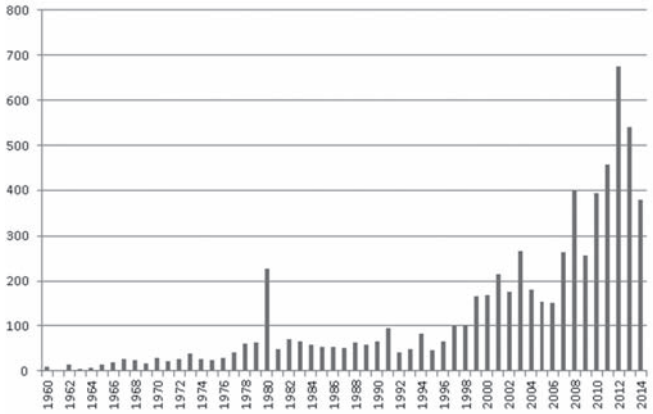


Main area distribution for zbMATH reviewers according to MSC

The ideal moment to become a reviewer is usually in the postdoc phase, where this activity comes with a broadening of mathematical views. Of course, it depends heavily on the choice of further career whether one is able to pursue reviewing for a longer time. There are a considerable proportion of reviewers who are able to share their experiences up to (and after) retirement (sometimes with a pause at the age where both scientific and administrative duties simply leave no time). While it is grossly unfair to pick out single names, we would like to mention some examples of reviewer longevity: for instance, Johann Jakob Burckhardt (famous for his work in group theory, crystallography and history of mathematics) wrote his first review in 1939 and his last review in 2004

at the age of 101 years; Janos Aczel started right after World War II in 1946 (when he obtained the auspicious reviewer number 007) and has written his last review (so far) in 2011; and, among the most dedicated zbMATH reviewers, we would also like to mention Grigori Mints, the great Russian logician, who started reviewing in 1970, when he was a young scholar at Leningrad State University, and continued for more than 500 reviews, also after moving to the US to take up a professorship at Stanford University. All the time, he critically accompanied new developments in logic and commented upon them, until he passed away too early on 29 May 2014, at an age of only 74. His enduring legacy will not only be his invaluable contributions to mathematical logic and its applications to philosophy and computer science but also the most dedicated reviews he wrote for zbMATH.

Of course, the circumstances of a scientific career often do not allow for such longevity; indeed, it seems that the increased pressure, especially on the younger generation, leads to time restrictions, which only allows for limited reviewing activity. On average, an active reviewer will be able to complete about three reviews a year, significantly less than in the past. The only way to balance this, and to secure a sufficient pool of experts, is the steady acquisition of new reviewers; indeed, there has been significant growth over the last few years, perhaps triggered by the now established custom of exchanging opinions via the web.[1]



Active zbMATH reviewer structure in terms of years of duty (with a starting point being the paper register of 1960)

This growth has also led to a massive internationalisation of the community. While there is still a domination of countries with a long mathematical tradition (more than 1,000 reviewers are located in both the US and Germany, with France, Italy, China and Russia closely following), there is now barely any country in the world without a mathematician taking up reviews (with a few exceptions like North Korea).

In the past, a driving force for taking up this activity has been the opportunity to access the newest research literature (this has been especially true, for example, in Eastern Europe during the Cold War, when material sent out for reviewing was sometimes the only source of Western publications, which led to a still thriving community in this region). Fortunately, this situation has changed. Also, the reviewer bonus (now € 5.12 per review when used to acquire Springer books with the 50% reviewer discount) is certainly now more a symbol of gratitude (indeed, nowadays, more and more reviewers use the opportunity to support the book donation program of the EMS for developing countries with their bonus). Hence, without doubt, the main motivation to contribute comes from the scientific advantages for the community outlined above and the common goal of ensuring the quality of mathematical research through a viable form of communication, which has grown into a reliable and persistent information infrastructure over many decades.

Olaf Teschke [teschke@zblmath.fiz-karlsruhe.de] is member of the Editorial Board of the EMS Newsletter, responsible for the Zentralblatt Column.

[1] New reviewers are usually selected and invited by the subject editors, but there is also the possibility to volunteer at https://zbmath.org/become-a-reviewer/

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Book Reviews

The Abel Prize 2008–2012

Holden and Piene (eds)

Springer Verlag
ISBN 978-3-642-39448-5
571 pp

Reviewer: Ulf Persson

This is the sequel to the first book on the Abel Prize winners, which covered the first five years and which was reviewed for the EMS a couple of years ago. The format of this book is the same but this time, even more ambitiously, the editors have produced a tome that dwarfs the preceding one in bulk. One wonders whether this will set a trend. The basic elements consist of autobiographical sketches followed by lengthy presentations of the mathematical accomplishments of the winners. In addition, there are complete lists of each winner's publications and short, formal CVs. There is also some additional material, which we will consider now and return to later.

The book starts out with an introduction, written by an historian Kim Helswig, presenting in documentary detail the history of how the prize came into being, a process in which the Abel biographer Arild Stubhaug played a key role both as an initiator and, perhaps more importantly, as a caretaker, making sure it came to fruition. There is also an account of the subsequent failure to connect it more closely with the Nobel Prize. In fact, the desired connection to the Nobel Prize is also made explicit by the title of the contribution. As was already discussed in the previous review, the Abel Prize is meant to be the Nobel Prize of mathematics, with the hope of also allowing mathematics to partake in the glamour and attention that comes to the natural sciences once a year. As was noted back then, it is one thing to set up a prize but quite another to make it famous. Financial generosity, as displayed by the Norwegian Government, may not hurt but it is far from being sufficient. Tradition is something of the past over which we have no control; it is different with the future for which we can always hold out hope. One can see these volumes as part of a sustained effort to establish the Abel Prize, at least in the world of mathematicians. The glamour of a prize never stems from the size of the prize itself, only from that of its recipients.

The cursory reader may be expected to read through the introduction and sample the autobiographical sketches for their human interest. Such a reader may also have the ambition to read through the mathematical sections (at least at some later date) but may find themselves bogged down in technicalities. It is indeed in the mathematical sections that we find the explanation for the added bulk; when it comes to autobiographies, the efforts are bound to be very sketchy and I do not believe that this will change in the future, as it is obviously not a requirement with which most of the mathematicians will feel comfortable. Thompson appears most uncomfortable; I suspect his one page submission, including a lengthy quote from Stendhal, must have been an effort for the editors to extricate. In the case of Tits, the interview reports serve the purpose; we learn that he must have been something of a prodigy, which may not have been revealed had he written a biography himself. Mathematicians tend to be modest and when it comes to autobiographies, this is a definite disadvantage. Tate and Milnor give succinct and impeccable accounts of their lives, especially their mathematical childhood and youth, which of course is of the greatest interest. We learn that Tate originally intended to be a physics graduate student at Princeton (although his interest and ability were definitely superior in mathematics) because from Bell's classic book 'Men of Mathematics' he had received the impression that you needed to be a genius to pursue mathematics (while, on the other hand, his father was a physicist). Milnor emphasises his shyness as a youngster and his consequent isolation, unsurprisingly taking advantage of any books he could lay his hands on, including the few mysterious mathematics books his father, an engineer, happened to own. When he came to Princeton at the age of 17, he was truly captivated and a subsequent stint in Zürich with Hopf as a graduate student capped it all off, not only mathematically. He also admits to a passing interest in game theory (after all, he was a fellow student of Nash) but decided that its main difficulties were not mathematical. The account of Szmeredi is not devoid of charm either but he plays it safe by concentrating on his early mathematical career. Gromov takes the most original approach by turning the task into a reflection on what it means to be a mathematician, in the process eschewing any systematic chronological account. The result is one of the gems in the volume. Scientists, like children, are good at non-understanding, he points out, in particular in their propensity for asking stupid questions, such as whether four elephants can beat a whale in a fight. One should never despise the trivial observation, he cautions, and refers to a chance remark in a lecture that made him realise that group theory was more than just slippery formalism, the consequences of which it would take him 20 years to work out. Mathematics is about asking the right questions and asking stupid questions is the way to start, he seems to imply. Gromov's parents were pathologists and the breadth of his interests is legendary; one surmises that he keeps on asking questions regardless of the context he finds himself in.

The meat of the book is to be found in the mathematical surveys – these differ widely. The most conventional, in a way, and this is not meant to be disparaging, is that

of Milne on Tate. Tate is an elegant mathematician, many of whose seminal insights remained notoriously unpublished; nevertheless, he exerted a deep influence in his field. This, incidentally, may be seen as an indication that too much is being published in mathematics. Similarly, Milne gives an elegant presentation of Tate's contributions, starting with his thesis under Emil Artin in class field theory, leading him into Abelian varieties, especially elliptic curves and their cohomology theory, heights, reductions and so on. Many conjectures are associated to his name, especially the Tate conjecture, with its connections both to Hodge and Birch-Swinnerton-Dyer. As an elementary example, one should recall his observations that a classical elliptic curve can also be presented as for some non-real q, which also makes sense in the p-adic setting, where one can no longer involve lattices. Beautiful formulas ensue, relating q to the j-invariant. All in all, Milne manages to convey the unified vision that underlies it all. He also provides, as a former student, some interesting remarks on the mathematical obiter dicta of Tate; it is a pity they are hidden in a footnote.

In the case of Milnor, no single contributor has been entrusted with the task of doing him justice. Bass writes on his algebraic contributions, such as his of K-theory (until Quillen came up with a definition of the higher K-groups in 1972, the definitions appeared quite ad hoc) and connections with quadratic forms. Lyubich writes about Milnor's later interest in dynamics, with many fractal pictures in full colour. Finally, Siebenmann writes about what one thinks of as Milnor's core interest: topology. To the general mathematical audience, he is associated with the exotic 7-sphere that exploded as a bomb in the mid 1950s and a few years later carried him to Stockholm (i.e. to ICM 1962). All in all, at 150 odd pages, roughly twice as long as that provided by Milne, it would make up a book by itself. Together, they provide an embarrassment of riches for the prospective reader to sample and savour. But, of course, to this is added another 100 pages covering Gromov's work. Here, nine authors contribute ten sections. One is reminded of blind men touching an elephant, some getting hold of the trunk, one of the tail, another one of a flapping ear (some unfortunate may be trampled by a foot). Appropriately enough, it is entitled 'A Few Snapshots from the Work of Mikhail Gromov'. These sections differ widely: many are anecdotal and some are more technical but they all adhere to the spirit of Gromov's own autobiographical sketch. As with the other surveys, it would be pointless to give an enumeration of topics. Basically, it is geometrical with the viewpoint of not only Riemannian geometry but also symplectic, with key concepts such as the h-Principle and the Waist inequality. As Cheeger noted during his presentation of Gromov's work during the Abel lectures at the time, it seems that half of what is known in differential geometry is known only to Gromov.

Thompson came onto the mathematical scene in the 1950s concomitantly with the revival of classical finite group theory, which would eventually result a quarter of a century later in the classification of finite simple groups. Most of us are familiar with elementary group theory including the existence of Sylow subgroups and have, no doubt, taken pleasure in the construction of groups of order say pq and being both amazed at and delighted by how much can be accomplished by so little. Although it is not normally thought of as part of combinatorics, it is an eminent example of such, and one which tends to appeal also to people who are not of a combinatorial temperament because it concerns individual objects with intricate structures. At the turn of the century, it had reached a high state of sophistication with the theory of group representations and characters, and the seminal works of Burnside and his conjectures. Thompson came onto the scene together with Feit in the early 1960s by showing that every simple group is of even order (thus every group of odd order is solvable), which set a precedent for long and intricate papers in the field. This is taken as a point of departure for the report on Thompson's work. With the kind of long and involved chains of combinatorial reasoning that goes with group theory, and which seldom can be conceptually summarised, such a survey cannot hope to give anything more than a taste for the subject. Tits' work also connects with finite groups (after all it was a shared award) but he also goes beyond it. As a starter, we can consider a problem he encountered and solved at the age of 16. We all know that the Moebius group acts triply transitively on the Riemann sphere. But this works not only for the complex numbers but for any field, in particular finite fields. What if we have a finite group acting triply transitively? Does it occur in that way, i.e. acting projectively on the projective line over a finite field? This is not quite so simple but if we add that the stabiliser of two points is commutative, mimicking the 1-parameter subgroup in the classical case, it goes through. To understand a group we need to see it in action. One may radically summarise the interest of Tits as providing geometrical structures on which groups are made to act – thus the reverse of the Erlangen programme. One of his more mature challenges was to give suggestive geometric interpretations to the exceptional Lie groups. Out of this, his elaborate theory with characteristic real estate terminology has evolved.

Szmeredi may be seen as the odd man out in the company, working in a less established field than the others, a field in which grand theories do not play an important role. It is probably for this reason that Gowers takes pains to point out, at the very end of his report, that Szmeredi is indeed worthy of the prize, something no other contributor feels it necessary to point out in relation to their own charge. Nevertheless, he works in an old tradition, represented by Erdos, with whom he actually has joint papers. Incidentally, looking at the list of publications of Szmeredi, one finds that nine out of ten are joint publications, no doubt following in the footsteps of his compatriot. He may be mostly known for his results that sets of positive density contain arithmetic sequences of arbitrary lengths, initially a conjecture of Erdos and Turan in the 1930s that stayed open for 40 years. This is combinatorics of quite a different nature from that of Thompson and Tits, pursued by bare hands. There are, of course, many other things he

has done and Gowers claims only to touch upon a few in his survey.

In addition to the presentations of the work and personalities of the Abel Prize winners, a digression is presented at the end, namely a facsimile of a letter Abel wrote to Crelle in the early Autumn of 1828 (he would succumb to tuberculosis the following April), a letter which was rediscovered by Mittag-Leffler just after the turn of the century and which sheds light on the relationship of Abel to the work of Galois. The letter indicates that Abel had a rather complete theory of equations which he was about to write down and have published, as soon as he had gotten the business with elliptic functions worked out. The latter seemed a less daunting task but ill health would soon deprive him of the opportunity to tackle the former. Christian Skau puts the letter, and ultimately the relationship between Abel and Galois, in an historical perspective, emphasising how closely Abel's work on elliptic functions was motivated by his interest in equations. Abel did not only show the impossibility of solving the 5th degree equation, he gave a sufficient and general condition for solvability, which is the reason the word abelian has become a synonym for commutative, something no mathematician should be unaware of. What would have happened if Abel and Galois had been graced with more normal life spans? Skau poses the obvious question about a contrafactual past. I suspect that similar digressions will be planned for future volumes, further widening their educational ambitions. One should never forget the legacy of Abel.

It is also a reviewer's duty to point out typographical errors, if for no other reason than to show that they have read through the book. The only typo I was able to spot was on page 530 where there are, on two occasions, a '–1' instead of a '+1'. But this is trivial and nothing to dwell on so let us instead not forget Abel's injunction to the effect of only studying the masters and not to be content with their followers, the exact quote of which is presented both on the flyleaf and at the end of Skau's contribution. Thus, one surmises, the ultimate purpose of the surveys is to send the reader back to the original sources.

For a picture and CV of the reviewer see p. 54 of this Newsletter.

Alexandre Grothendieck. A Mathematical Portrait

Leila Schneps (Ed.)

International Press of Boston
ISBN 978-1571462824
330 pp

Reviewer: Jean-Paul Allouche

The importance of the work of Alexandre Grothendieck makes him undoubtedly one of the most important mathematicians of the 20th century. This book, as noted in the introduction, is partially the outcome of a conference organised in 2008 by P. Lochak, W. Scharlau and L. Schneps, entitled "Alexandre Grothendieck: Biography, Mathematics, Philosophy". During this conference, a group of participants discussed the possibility and necessity of writing a book explaining to beginners in mathematics the extraordinary importance of Grothendieck's work. Whether this book is a vulgarisation/popularisation of his work depends on how familiar the reader already is with some parts of mathematics. Though it can probably not be read – even by a mathematician – like a novel, it has great merit in trying to be readable, provided the reader makes some effort to enter into this fascinating world. We will certainly not try to give a detailed account of the book in just these few pages. Rather, we hope that we will succeed in teasing the reader into first reading the book and then tackling one (or more) of Grothendieck's papers.

The first paper, by J. Diestel, is about the contribution of Grothendieck to functional analysis and, more precisely, to Banach space theory in five papers that appeared between 1953 and 1956. The author writes about one of Grothendieck's papers on the subject: "This groundbreaking paper continues to be a source of inspiration; a list of mathematicians who have improved on its results reads like a 'Who's Who' of modern functional analysis." Interestingly, the author indicates that, though Grothendieck was certainly a "theory builder", he often "solved problems" while building general structures.

The next paper, by M. Karoubi, shows the influence of Grothendieck in K-theory (for the connoisseurs, we just mention the then mysterious group $K(X)$, introduced by Grothendieck in 1957 in his proof of the Riemann–Roch theorem). On a personal side, the author alludes to the generosity of Grothendieck, who was always ready to give mathematical advice.

In the third paper, M. Raynaud writes about Grothendieck and the theory of schemes. The author insists that Grothendieck always chose the most general framework, using in particular the language of categories. We cannot resist writing a sentence here from *Récoltes et Semailles* that M. Raynaud quotes in the paper: "… je n'ai pu m'empêcher, au fur et à mesure, de construire des maisons, des très vastes et des moins vastes, et toutes bonnes à être habitées, – des maisons où chaque coin et recoin est destiné à devenir lieu accueillant et familier pour plus d'un. Les portes et fenêtres sont d'aplomb et s'ouvrent et se ferment sans entrebâiller et sans grincer, le toit ne fuit pas et la cheminée tire."

Much more about schemes can be found in the next two papers: *The Picard scheme* by S.L. Kleiman and *My introduction to schemes and functors* by D. Mumford. In these two papers the fabulous influence of Grothendieck appears again. It seems almost trivial to underline this point but the fact that he had a tremendous influence on mathematics is more and more extraordinarily visible while reading the book, even if one already knows (part of) this influence.

The short title of the sixth paper, *Descent* by C.T. Simpson, corresponds to the longest paper of the book. The reader will enthusiastically see how a notion that, roughly speaking, dates back to medieval cartography underlies modern notions (cohomology, topos, sheaves, stacks, etc.).

The next paper, by J.P. Murre, describes the work of Grothendieck on the (algebraic) fundamental group. We will simply note that the author briefly alludes at the end of the paper to one of the last papers of Grothendieck (which remained unpublished from 1984 to 1997 and was actually not intended to be a mathematical paper): *Esquisse d'un Programme*.

The eighth paper, *An apprenticeship*, by R. Hartshorne, alludes to deep mathematical questions while also giving personal memories about Grothendieck. What we would like to note here is that the author concludes his paper by saying he was puzzled and confused when Grothendieck decided to stop doing mathematics; he writes in particular: "For me, he remains unknown and unknowable by any 'normal' standards of human psychology and behaviour."

The next two papers are a paper by L. Illusie on Grothendieck and the étale cohomology, where the author, in particular, writes that Grothendieck probably had the idea of introducing étale cohomology after a talk of Serre at a Chevalley seminar in 1958, and a paper by L. Schneps devoted to the influential Grothendieck–Serre correspondence. It might well be that the latter paper could be (one of) the first to read as a deep motivation for becoming more familiar with Grothendieck's work and appreciating the rest of the book. No short abstract of the paper of L. Schneps can be reasonably given: we urge the reader to look at it, for both the mathematical and the personal aspects of Grothendieck's work.

A very important point, addressed in the 11th paper *Did earlier thoughts inspire Grothendieck?* by F. Oort, addresses the question of how Grothendieck achieved so many fundamental ideas in what sometimes seems "black magics" (Munford). Oort gives some very interesting sources for Grothendieck's inspiration but he also gives a kind of warning: "Every time I started out expecting to find that a certain method was originally Grothendieck's idea in full, but then, on closer examination, I discovered each time that there could be found in earlier mathematics some preliminary example, specific detail, part of a proof, or anything of that kind that preceded a general theory developed by Grothendieck. However, seeing an inspiration, a starting point, it also showed what sort of amazing quantum leap Grothendieck did take in order to describe his more general results or structures he found."

The penultimate and final papers are *A country of which nothing is known but the name: Grothendieck and 'motives'* by P. Cartier and *Forgotten motives: the variety of scientific experience* by Y.I. Manin. Both papers speak about motives. Both give hints to Grothendieck's life. But while Cartier suggests a very naive approach (he writes that he is "an absolute novice in the domain of psychoanalysis") of Grothendieck's "fundamental wound" by the "absent father", Manin writes "Thinking back on his imprint on me then, I realize that it was his generosity and his uncanny sense of humor that struck me most, the carnivalistic streak in his nature, which I later learned to discover in other anarchists and revolutionaries".

What strikes me in the view that mathematicians have of Grothendieck's work and life is the huge gap between the interest and fascination for his mathematical work and – at least for a large majority of mathematicians – the fact that they are rejecting the rest of Grothendieck's thoughts (should we accept military grants, the unbearable explicit hierarchy that mathematicians build among themselves, the urgency of working in ecology and the like instead of dealing with mathematics, etc.). It is so easy to declare that he was "a bit mad" or "deeply depressed" or even "suffering of psychosis" rather than to think that he could well have been very much in advance also in these subjects. A related point is that the mathematical community seems to have the desire to publish his unpublished works, though he seems to have strongly demanded explicitly that this should not be the case and that everything should be destroyed: is it normal that the last will of anyone, a fortiori a very important mathematician, is not respected?

Erratum

In the last issue of the Newsletter (Issue 94, December 2014), one of the books on quantum theory suggested by H. Nishimura, the author of the review of *Quantum Theory for Mathematicians* by Brian C. Hall (pages 55–56), has been inadvertently omitted: E. Zeidler, Quantum field theory. I: Basics in mathematics and physics. A bridge between mathematicians and physicists (https://zbmath.org/?q=an:1124.81002).

Solved and Unsolved Problems

Themistocles M. Rassias (National Technical University, Athens, Greece)

We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds, so we cannot completely prescribe its properties a priori.

Carl Friedrich Gauss (1777–1855),
Letter to Bessel, 1830

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

139. If we define $T_{n,r}^{\rho}(x)$ with $n \in \mathbb{N}, r \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ and $\rho > 0$ as follows:

$$T_{n,r}^{\rho}(x) = \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} b_{n,k}^{\rho}(t) t^r \, dt,$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$
$$b_{n,k}^{\rho}(t) = \frac{1}{B(k\rho, n\rho+1)} \frac{t^{k\rho-1}}{(1+t)^{k\rho+n\rho+1}},$$

prove that for $n\rho > r$, the following recurrence relation holds:

$$\left(n - \frac{r}{\rho}\right) T_{n,r+1}^{\rho}(x) = x(1+x)[T_{n,r}^{\rho}(x)]' + \left(\frac{r}{\rho} + nx\right) T_{n,r}^{\rho}(x).$$

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

140. If we define $T_{n,r}(x)$ with $n \in \mathbb{N}$ and $r \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ as follows:

$$T_{n,r}(x) = \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} b_{n,k}(t) t^r \, dt,$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$
$$b_{n,k}(t) = \frac{1}{B(k, n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}},$$

prove that

$$T_{n,r}(x) = \frac{\Gamma(n-r+1)\Gamma(r+1)x}{\Gamma(n)} \; {}_2F_1(n+1, 1-r; 2; -x)$$

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

141. For a real number $a > 0$, define the sequence $(x_n)_{n\geq 1}$, $x_n = \sum_{k=1}^{n} a^{\frac{k^2}{n^3}} - n$.
(1) Prove that $\lim_{n\to\infty} x_n = \frac{1}{3}\ln a$.
(2) Evaluate $\lim_{n\to\infty} n(x_n - \frac{1}{3}\ln a)$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

142. Let $a < b$ be positive real numbers. Prove that the system

$$\begin{cases} (2a+b)^{x+y} = (3a)^x (a+2b)^y \\ (a+2b)^{y+z} = (2a+b)^y (3b)^z \end{cases}$$

has a solution (x, y, z) such that $a < x < y < z < b$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

143. A function $f : I \subset \mathbb{R} \to (0, \infty)$ is called *AH-convex (concave)* on the interval I if the following inequality holds

$$f((1-\lambda)x + \lambda y) \leq (\geq) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda \frac{1}{f(y)}} \qquad \text{(AH)}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.
Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. Show that if $a, b \in I$ with $a < b$ then we have the inequality

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) \left[\frac{b-s}{b-a} f(b) + \frac{s-a}{b-a} f(a)\right] f(s) \qquad (1)$$

for any $s \in [a, b]$.
In particular, we have

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) f\left(\frac{a+b}{2}\right) \frac{f(a)+f(b)}{2} \qquad (2)$$

and

$$\frac{1}{b-a} \int_a^b f^2(t)\, dt \leq (\geq) f(a) f(b). \qquad (3)$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

144. Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a, b]$. Show that, if $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequality

$$\Phi\left(\frac{(s-a) f(a) + (b-s) f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\, dt\right)$$
$$\leq (\geq) \frac{s-a}{(b-a)^2} \int_a^b \Phi[f(a) - f(t)]\, dt + \frac{b-s}{(b-a)^2} \int_a^b \Phi[f(b) - f(t)]\, dt \qquad (4)$$

for any $s \in [a, b]$.
In particular, we have

$$\Phi\left(\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)\, dt\right)$$
$$\leq (\geq) \frac{1}{b-a} \int_a^b \frac{\Phi[f(a) - f(t)] + \Phi[f(b) - f(t)]}{2} dt. \qquad (5)$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

II Two new open problems

145*. Compute the following limit

$$\lim_{k\to\infty} \frac{(1+\sin\alpha_{k+1})^{\alpha_{k+1}}}{(1+\sin\alpha_k)^{\alpha_k}}$$

on

$$\{\alpha_k \in \mathbb{N} \mid k \in \mathbb{N} : \sin\alpha_k > \sin\alpha_{k-1}\} = \{1, 2, 8, 14, 33, 322, 366, 699, \ldots\},$$

that is, numbers n such that $\sin(n)$ increases monotonically to 1.

(*Pierluigi Vellucci, SBAI – Sapienza Università di Roma*)

146*. Given a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

with $a_0, a_1, \ldots, a_n$ given real (or complex) numbers with $a_0 \neq 0$, examine whether one can determine the sum of its real roots only in terms of the coefficients of the polynomial.

(*E. Kyriakopoulos, National Technical University of Athens, Athens, Greece*)

III Solutions

131. Let $ABCD$ be an isosceles trapezium with $AB = \frac{3}{2}$, angle $CAD = 90^\circ$ and angle $CAB = 15^\circ$. On the same part of the line CD with A and B we consider a point M such that $MC = \sqrt{2}$ and $MD = 1$. Find $MA + MB$.

(*Cristinel Mortici, Valahia University of Târgovişte, Romania*)

Solution by the proposer. The circle Ω of diameter CD passes through A, B, C, D.

In ΔAOB ($OA = OB = \frac{DC}{2}$, $AB = \frac{3}{2}$, $\measuredangle AOB = 120^0$), we have $OA = OB = \frac{\sqrt{3}}{2}$ and then $CD = \sqrt{3}$.

$CD^2 = MD^2 + MC^2$, thus $M \in \Omega$.

$AD < MD$, $BC < MC$, therefore M lies on the small arc AB.

Let $T \in \Omega$ be the middle point of the arc CD.

Ptolemeu's theorem in $MATB$ (ΔATB equilateral) gives

$$MA + MB = MT.$$

Ptolemeu's theorem in $MDTC$ (ΔDTC right-angled) gives

$$MC + MD = MT\sqrt{2}.$$

Now

$$(MA + MB)\sqrt{2} = MC + MD,$$

so

$$MA + MB = 1 + \frac{\sqrt{2}}{2}.$$

□

Also solved by Tim Cross (King Edward's School, Birmingham, UK), Andrea Fanchini (Cantu, Italy), Demitrios Koukakis (Kilkis, Greece), Panagiotis T. Krasopoulos (Athens, Greece), Nikolaos Pavlou (Thessaloniki, Greece)

132. Consider the regular polygon $A_1A_2\ldots A_n$. Find the minimum and the maximum of the lengths of the segments with ends on the sides of the polygon and passing through its centre.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*)

Solution by the proposer. For n even, because of the symmetry of the polygon, the minimum is attained by the segments passing through the midpoints of two opposite sides and the maximum by the segments passing through two opposite vertices. In this case, the value of the minimum is $2R\cos\frac{\pi}{n}$ and the value of the maximum is $2R$, where R is the circumradius of the polygon.

For n odd, consider O the centre of the polygon. In this case, A_1O is the symmetry axis of the polygon. Let M be the intersection point of the line A_1O with the side opposite to the vertex A_1 and let N be a vertex of this side. It suffices to determine the minimum and the maximum of the length of the variable segment AB, passing through O and intersecting MN in B. Note that

$$AB = AO + OB = OM\left(\frac{1}{\cos\widehat{MOB}} + \frac{1}{\cos\widehat{BON}}\right).$$

We have $OM = R\cos\frac{\pi}{n}$ and $\widehat{BON} = \frac{\pi}{n} - \widehat{MOB}$, hence the problem is reduced to finding the extremal values of the function

$$f(x) = \frac{1}{\cos x} + \frac{1}{\cos(a-x)},$$

where $a = \frac{\pi}{n}$ and $x \in [0, a]$. We have

$$f'(x) = \frac{\sin x}{\cos^2 x} - \frac{\sin(a-x)}{\cos^2(a-x)}$$

and

$$f''(x) = \frac{1+\sin^2 x}{\cos^3 x} + \frac{1+\sin^2(a-x)}{\cos^3(a-x)}.$$

Observe that $f'(\frac{a}{2}) = 0$ and $f''(x) > 0$ for any $x \in [0, a]$, that is, f' is strictly increasing. It follows that f' is negative on the interval $[0, \frac{a}{2})$ and positive on $(\frac{a}{2}, a]$. Therefore, $x = \frac{a}{2}$ is the unique minimum point of f. We also have $f(0) = f(a) = 1 + \frac{1}{\cos a}$, giving the maximum value of function f.

In this case the minimum is $2R\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}$ and the maximum is $R(1+\cos\frac{\pi}{n})$. □

Also solved by Alberto Bersani (Sapienza Università di Roma Italy), M. Bencze (Brasov, Romania), Demitrios Koukakis (Kilkis Greece), Sotirios E. Louridas (Athens Greece)

133. In the convex quadrilateral $ABCD$, consider P the intersection point of its diagonals and E, F the projections of P on the sides AB and CD. Let M, N be the midpoints of the sides BC and AD. Prove that if $MN \perp EF$ then the quadrilateral $ABCD$ is cyclic or a trapezoid.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*)

Solution by the proposer. We introduce the following notation: $\widehat{APB} = \widehat{CPD} = \pi - \alpha$, $\widehat{PAB} = x$, $\widehat{PBA} = \alpha - x$, $\widehat{PDC} = y$ and $\widehat{PCD} = \alpha - y$, and we then have

$$\overrightarrow{EF}\cdot\overrightarrow{MN} = (\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{PN}-\overrightarrow{PM}) = \frac{1}{2}(\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{PB}+\overrightarrow{PC}-\overrightarrow{PA}-\overrightarrow{PD})$$

$$= \frac{1}{2}(\overrightarrow{PF}-\overrightarrow{PE})\cdot(\overrightarrow{AB}+\overrightarrow{DC}) = \frac{1}{2}(\overrightarrow{PF}\cdot\overrightarrow{AB} - \overrightarrow{PE}\cdot\overrightarrow{DC}).$$

It follows that $\overrightarrow{EF} \cdot \overrightarrow{MN} = 0$ if and only if $\overrightarrow{PF} \cdot \overrightarrow{AB} = \overrightarrow{PE} \cdot \overrightarrow{DC}$. But it is clear that we have $(\widehat{\overrightarrow{PF}, \overrightarrow{AB}}) = (\widehat{\overrightarrow{PE}, \overrightarrow{DC}})$, hence we obtain $PF \cdot AB = PE \cdot CD$. The last relation is equivalent to

$$\frac{2\sigma[CPD]}{CD} \cdot AB = \frac{2\sigma[APB]}{AB} \cdot CD$$

and we get

$$\left(\frac{AB}{CD}\right)^2 = \frac{\sigma[APB]}{\sigma[CPD]} = \frac{AP \cdot PB}{CP \cdot PD}.$$

Using the law of sines in triangles APB and CPD, the last relation is equivalent to

$$\frac{\cos\alpha}{\sin x \sin(\alpha - x)} = \frac{\cos\alpha}{\sin y \sin(\alpha - y)},$$

that is, $\sin x \sin(\alpha - x) = \sin y \sin(\alpha - y)$. From this relation we get $\cos(2x - \alpha) - \cos\alpha = \cos(2y - \alpha) - \cos\alpha$, hence $\cos(2x - \alpha) = \cos(2y - \alpha)$. It follows that $-2\sin(x + y - \alpha)\sin(x - y) = 0$, hence $x = y$ or $x = \alpha - y$. In the first case, we obtain that the quadrilateral $ABCD$ is cyclic. From the relation $x = \alpha - y$, it follows that the quadrilateral $ABCD$ is trapezoid. □

Also solved by M. Bencze (Brasov, Romania), Tim Cross (King Edward's School, Birmingham, UK), Demitrios Koukakis (Kilkis, Greece), Sotirios E. Louridas (Athens, Greece)

Einstein Addition: Background. Until recently, the general Einstein addition of relativistically admissible velocities that need not be parallel has rested in undeserved obscurity. The following problems uncover the rich mathematical life that Einstein addition possesses.

Let $c > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n = 1, 2, 3, \ldots$, equipped with common vector addition "+" and inner product "·". The home of all n-dimensional Einsteinian velocities is the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}. \tag{6}$$

The c-ball $\mathbb{R}^n_c$ is the open ball of radius c, centred at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation "⊕" in the c-ball $\mathbb{R}^n_c$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1 + \gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{7}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_\mathbf{v}$ is the gamma factor

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \geq 1 \tag{8}$$

and where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. Einstein subtraction is denoted by $\ominus\mathbf{v} = -\mathbf{v}$, so that $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$. The pair $(\mathbb{R}^n_c, \oplus)$ forms the n-dimensional Einstein gyrogroup.

134.

(1) Prove that Einstein addition "⊕" and the gamma factor "$\gamma_\mathbf{v}$" are related by the *gamma identity*

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{9}$$

(2) Prove that Einstein addition is closed in the ball $\mathbb{R}^n_c$, that is, prove that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ then $\mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^n_c$.

(*Abraham A. Ungar, Department of Mathematics, North Dakota State University, USA*)

Solution by the proposer: Solution to Problem 134(1): The gamma identity (9) is equivalent to the equation

$$\gamma^2_{\mathbf{u}\oplus\mathbf{v}} = \gamma^2_\mathbf{u}\gamma^2_\mathbf{v}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^2. \tag{10}$$

This equation, in turn, can readily be proved by lengthy but straightforward algebra. The use of computer software, like Mathematica or Maple, for symbolic manipulation is recommended.

Solution to Problem 134(2): Clearly, $\mathbf{v} \in \mathbb{R}^n_c$ if and only if the gamma factor $\gamma_\mathbf{v}$ of $\mathbf{v}$ is real. It is also clear from the gamma identity (9) that if $\gamma_\mathbf{u}$ and $\gamma_\mathbf{v}$ are real then $\gamma_{\mathbf{u}\oplus\mathbf{v}}$ is real. Hence, if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ then $\mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^n_c$. □

Also solved by Jim Bradley (North Patchole Farmhouse, Devon, UK), Soon-Mo Jung (Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece)

135. Prove that Einstein addition "⊕" obeys the *Einstein triangle inequality*

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{11}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

(*Abraham A. Ungar, Department of Mathematics, North Dakota State University, USA*)

Solution by the proposer. When the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{R}^n_c$ of $\mathbb{R}^n$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, Einstein addition (7) reduces to the Einstein addition of parallel velocities,

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}}, \qquad \mathbf{u}\|\mathbf{v}\,. \tag{12}$$

Following (12), we have

$$\|\mathbf{u}\|\oplus\|\mathbf{v}\| = \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 + \frac{1}{c^2}\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. By the gamma identity (9) and by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \gamma_{\|\mathbf{u}\|\oplus\|\mathbf{v}\|} &= \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1 + \frac{\|\mathbf{u}\|\|\mathbf{v}\|}{c^2}\right) \\ &\geq \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \\ &= \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ &= \gamma_{\|\mathbf{u}\oplus\mathbf{v}\|} \end{aligned} \tag{14}$$

for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n_c$. But $\gamma_\mathbf{x} = \gamma_{\|\mathbf{x}\|}$ is a monotonically increasing function of $\|\mathbf{x}\|$, $0 \leq \|\mathbf{x}\| < c$. Hence (14) implies

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{15}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. □

Also solved by Soon-Mo Jung (Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece)

A subset C of a metric space X is called a *Chebyshev set* if for each point of X there exists a unique nearest point in C. It is well known that every closed convex set in a finite dimensional Euclidean vector space is a Chebyshev set. The converse is also true and it was shown independently by L. N. H. Bunt (1934) and T. S. Motzkin (1935). The problem that we are going to state leads to an alternative proof of this fact for the case of a 2-dimensional Euclidean vector space.

The following notation will be used. By E_2 we denote the Euclidean plane. For $x, y \in E_2$ we write $[x, y] = \{(1-\lambda)x+\lambda y : 0 \le \lambda \le 1\}$ for the closed segment and $(x, y) = \{(1-\lambda)x+\lambda y : 0 < \lambda < 1\}$ for the open segment with endpoints x, y. The set $D = D(x_0, r) = \{y \in E_2 : \|y - x_0\| \le r\}$ is the closed disc with centre x_0 and radius $r > 0$. For a subset C of E_2 we denote by $conv(C)$ the convex hull of C. If X is a topological space and $Y \subseteq X$, we denote by $Int(Y)$ the *interior* of Y and by $Bd(Y)$ the *boundary* of Y. We will say that Y is a *retract* of X if there exists a continuous map $R : X \to Y$ such that $R(y) = y$ for every $y \in Y$.

136. Let C be a Chebyshev set in E_2.

(a) Show that for every closed disc $D = D(x_0, r)$ with $x_0 \in C$, the intersection $C \cap D$ is a Chebyshev set.

(b) Assume that C is a bounded set and let $K = \text{conv}(C)$. If the boundary of K is contained in C, show that the sets K and C coincide.
(Hint: Use Brouwer's theorem.)

(c) Let x_1, x_2 be two distinct points of C. Also let $x_0 \in (x_1, x_2)$ and let ζ be the perpendicular line to $[x_1, x_2]$ that passes through the point x_0. If D is a closed disc containing x_1, x_2, show that $D \cap \zeta \cap C \neq \emptyset$.
(Hint: If $D = D(z_0, r)$ does not satisfy the conclusion, show that $\{x_1, x_2\}$ is a retract of the set $[x_1, z_0] \cup [z_0, x_2]$.)

(d) Prove that C is convex.
(Hint: Combine the above.)

(*Vassilis Kanellopoulos, National Technical University of Athens, Department of Mathematics, Greece*)

Solution by the proposer. Let $P_C : E_2 \to C$ be the metric projection of C (also called nearest point map), that is, $P_C(x)$ is the nearest point to x in C. It is well known that P_C is continuous.

(a) Let $x_0 \in C$, $r_0 > 0$ and $D = D(x_0, r_0)$. Without loss of generality, we may assume that $x_0 = 0$. Fix a point $x \in E_2$. If $P_C(x) \in D$ then clearly $P_C(x)$ is the nearest point of $C \cap D$. Otherwise, $\|P_C(x)\| > r_0$. Since $P_C(0) = 0$, by continuity of the map $\|P_C(x)\|$, there exists a $\lambda \in (0, 1)$ such that $\|P_C(\lambda x)\| = r_0$. The point $y := P_C(\lambda x)$ is the nearest to x point of $C \cap D$. Indeed, it is easy to see that $D(x, \|x - y\|) \cap D \subseteq D(\lambda x, \|\lambda x - y\|)$ and, thus, $D(x, \|x - y\|) \cap (D \cap C) = \{y\}$.

(b) Let us assume on the contrary that $K \neq C$ and choose a point $x_0 \in K \setminus C$. Since $Bd(K) \subseteq C$ we have that $x_0 \in Int(K)$. For every $x \in E_2 \setminus \{x_0\}$, denote by $\eta(x)$ the ray with origin x_0 that passes through the point x. Now, define the map $R : K \to Bd(K)$ by $R(x) = \eta(P_C(x)) \cap Bd(K)$. Notice that R is continuous and $R(x) = x$ for every $x \in Bd(K)$. Therefore, the boundary of K is a retract of K, which is a contradiction by Brouwer's theorem.

(c) Suppose on the contrary that there exists a closed disc $D = D(z_0, r)$ such that $x_1, x_2 \in D$ and $D \cap \zeta \cap C = \emptyset$. Without loss of generality, let $x_0 = 0$. Denote by ε the straight line determined by the points x_1, x_2 and let P_ε be the orthogonal projection of E_2 onto ε. Also, for every $\lambda > 0$ and $i \in \{1, 2\}$, set $P(\lambda x_i) = x_i$. We set $\Lambda = [x_1, z_0] \cup [z_0, x_2]$ and let $R : \Lambda \to \{x_1, x_2\}$ defined by $R(x) = P(P_\varepsilon(P_C(x)))$. Notice that since $x_1 \in D(z_0, r) \cap C$, we have $P_C(x) \in D(x, \|x - x_1\|) \cap C \subseteq D(z_0, r) \cap C$ for every $x \in [x_1, z_0]$. Similarly, $P_C(x) \in D(z_0, r) \cap C$ for every $x \in [z_0, x_2]$. Therefore, by our assumption that $D \cap \zeta \cap C = \emptyset$, we conclude that $P_C(x) \notin \zeta$, which yields that $P_\varepsilon(P_C(x)) \neq 0$ for all $x \in \Lambda$. Hence, the map R is well-defined. Moreover $R(x_i) = x_i$ for $i \in \{1, 2\}$. However, this means that the pair $\{x_1, x_2\}$ is a retract of Λ, which is a contradiction by the connectedness of Λ.

(d) Let C be a Chebyshev set in E_2. We will show that C is a convex set. By (a) we may assume that C is compact. Set $K = conv(C)$. By (b), it suffices to show that the boundary of K is contained in C. So fix $x_0 \in Bd(K)$ and let ε be the supporting line of K at x_0. If $\varepsilon \cap K = \{x_0\}$ then x_0 is an extreme point of K and so $x_0 \in C$. Otherwise, there exist two distinct points $x_1, x_2 \in K$ such that $\varepsilon \cap K = [x_1, x_2]$ and $x_0 \in (x_1, x_2)$. Since $[x_1, x_2]$ is a face of K, the points x_1, x_2 are extreme points of K and, thus, $x_1, x_2 \in C$. Let ζ be the perpendicular line to ε passing through x_0 and set $J = \zeta \cap K$. Notice that J is a bounded closed segment containing $\zeta \cap C$ and, since ε is a supporting line of K, x_0 is an endpoint of J. We have to show that $x_0 \in C$. Indeed, assuming that $x_0 \notin C$, it is possible to find a closed subsegment J' of J such that $x_0 \notin J'$ and $C \cap \zeta \subseteq J'$. Notice that $[x_1, x_2] \cap J' = \emptyset$. Hence, we have $[x_1, x_2] \subseteq \varepsilon$, $J' \subseteq \zeta$ and $\varepsilon \perp \zeta$. We can now easily see that there exists a closed disc D with $[x_1, x_2] \subseteq D$ and $D \cap J' = \emptyset$. But then $D \cap \zeta \cap C \subseteq D \cap J' = \emptyset$, which contradicts (b). Hence, $x_0 \in C$ and the proof of (d) is completed.

□

Also solved by M. Bencze (Brasov, Romania), Soon-Mo Jung (Chochiwon, Korea), Sotirios E. Louridas (Athens, Greece)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Real Analysis*.

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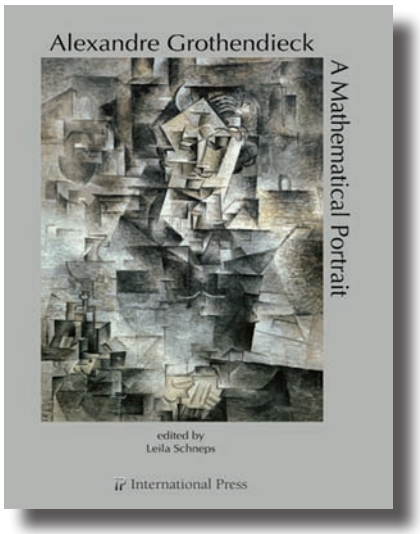
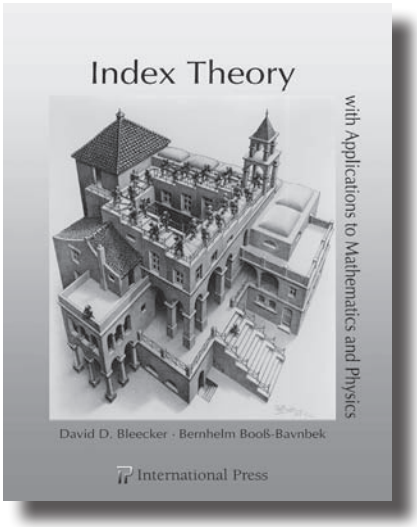
Alexandre Grothendieck: A Mathematical Portrait



Editor: Leila Schneps (Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, Paris)

"The book under review is irresistible to anyone who has even a mild interest in and acquaintance with algebraic geometry... and who is fascinated by Grothendieck's remarkable life. The authors represented in this "Mathematical Portrait" are uniquely positioned to comment not only on Grothendieck's mathematics but on the man himself, his personality, his influence (and his influence on them), and his uniqueness. Yes indeed, I think the book is altogether irresistible. —Prof. Michael Berg (for MAA Reviews) Loyola Marymount University

This book attempts to provide a reasonable explanation of what made Grothendieck the mathematician that he was. Thirteen articles written by people who knew him personally—some who even studied or collaborated with him over a period of many years—portray Grothendieck at work, explaining the nature of his thought through descriptions of his discoveries and contributions to various subjects, and with impressions, memories, anecdotes, and some biographical elements. Seeing him through the eyes of those who knew him well, the reader will come away with a better understanding of what made Grothendieck unique Subscription price includes shipping. Individual issues are $50 list price. For more information, please visit http://intlpress.com/ICCM. ***Languages: English, French * Hardcover * 324 pages plus 6 pages of photographs of contributing authors * Published June 2014 * isbn 9781571462824 * List price: $85***



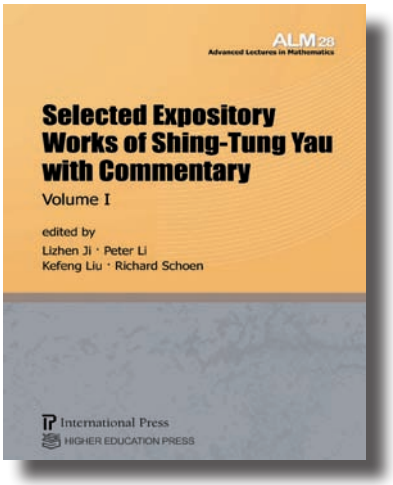
Index Theory with Applications in Mathematics and Physics

Authors: David D. Bleecker (University of Hawaii at Manoa) and Bernhelm Booß-Bavnbek (Roskilde Universitet, Denmark)

Professors Bleecker and Booss-Bavnbek have followed … developments in index theory from the beginning, and made original contributions of their own … Assuming only basic analysis and algebra, [this book] gives detailed constructions and proofs for all the necessary concepts, along with illuminating digressions on the various paths through the rich territory of index theory." —Robert Seeley, Professor Emeritus, University of Massachusetts, Boston

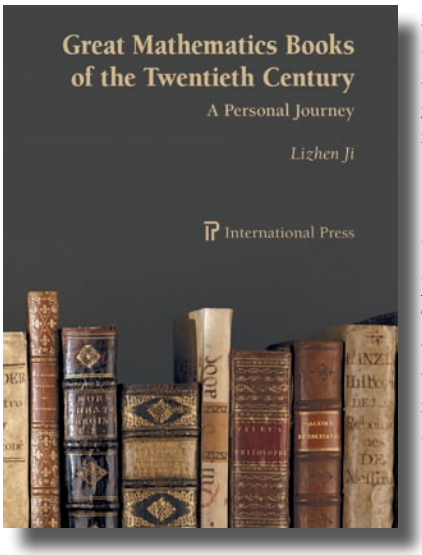
In this ambitious new work, authors David Bleecker and Bernhelm Booß-Bavnbek give two proofs of the Atiyah-Singer Index Theorem in impressive detail: one based on K-theory and the other on the heat kernel approach. The 18 chapters and two appendices of the book introduce different topics and aspects, often beginning "from scratch" without presuming full knowledge of all the preceding chapters. *Index Theory with Applications to Mathematics and Physics* is a textbook, a reference book, a survey, and much more. Written in a lively fashion, it contains a wealth of basic examples and exercises. The authors have included many discussion sections that are both entertaining and informative, which illuminate the thinking behind the more general theory. A detailed bibliography and index facilitate the orientation. ***Hardcover * 792 pages * Published Oct 2013 * isbn 978157146240 * List price $95***

Selected Expository Works of Shing-Tung Yau with Commentary



Editors: Lizhen Ji (University of Michigan), Peter Li (University of California at Irvine), Kefeng Liu (University of California at Los Angeles) and Richard Schoen (Stanford University)

Comprising volumes 28 and 29 of the ALM series, this outstanding collection presents all the survey papers of Shing-Tung Yau published to date (through 2013), each with Yau's own commentary. Among these are several papers not otherwise easily accessible. Also presented are several commentaries on Yau's work written by outstanding scholars from around the world especially for publication here. The 2-volume set provides the reader with systematic commentary on all aspects of mathematics by a contemporary master. The reader can thereby see the world of mathematics through his particular perspective, and gain understanding of the motivation and evolution of mathematical ideas. ***Hardcover * 1,418 pages * Published October 2014 * isbn 9781571462954 * List price $189***



Great Mathematics Books of the Twentieth Century: A Personal Journey

Editor: Lizhen Ji

The present volume serves as a guide to the most important books among the vast mathematical literature of the twentieth century, organized by subject. In doing so, it provides concise summaries of all major subjects in contemporary mathematics. Also included are brief one-paragraph introductions to numerous works of mathematics from throughout the ages, by old masters such as Euler, Galileo, Gauss, Kepler, Leibniz, Newton, Poincaré, et al, accompanied by many images of early printed editions. ***Hardcover * Published March 2014 * isbn 781571462831 * List price $119.***

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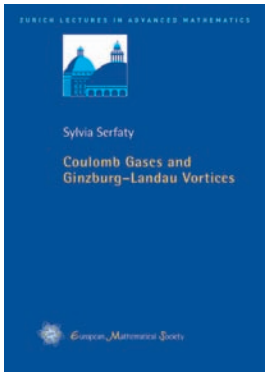
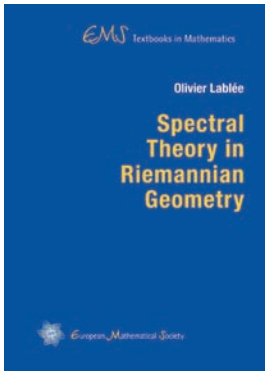


Sylvia Serfaty (Université Pierre et Marie Curie (Paris VI), France, and New York University, USA)
Coulomb Gases and Ginzburg–Landau Vortices (Zurich Lectures in Advanced Mathematics)

ISBN 978-3-03719-152-1. 2015. 168 pages. Softcover. 17 x 24 cm. 34.00 Euro

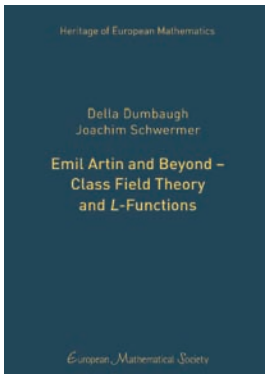
The topic of this book is systems of points in Coulomb interaction, in particular, the classical Coulomb gas, and vortices in the Ginzburg–Landau model of superconductivity. The classical Coulomb and Log gases are standard statistical mechanics models, which have seen important developments in the mathematical literature due to their connection with random matrices and approximation theory. At low temperature, these systems are expected to "cristallize" to so-called Fekete sets, which exhibit microscopically a lattice structure.
The Ginzburg–Landau model, on the other hand, describes superconductors. In super-conducting materials subjected to an external magnetic field, densely packed point vortices emerge, forming perfect triangular lattice patterns, so-called Abrikosov lattices.
This book describes these two systems and explores the similarity between them. It presents the mathematical tools developed to analyze the interaction between the Coulomb particles or the vortices, at the microscopic scale, and describes a "renormalized energy" governing the point patterns. This is believed to measure the disorder of a point configuration, and to be minimized by the Abrikosov lattice in dimension 2. The book gives a self-contained presentation of results on the mean field limit of the Coulomb gas system, with or without temperature, and of the derivation of the renormalized energy.



Olivier Lablée (Université Joseph Fourier Grenoble 1, Saint Martin d'Hères, France)
Spectral Theory in Riemannian Geometry (EMS Textbooks in Mathematics)

ISBN 978-3-03719-151-4. 2015. 197 pages. Hardcover. 16.5 x 23.5 cm. 38.00 Euro

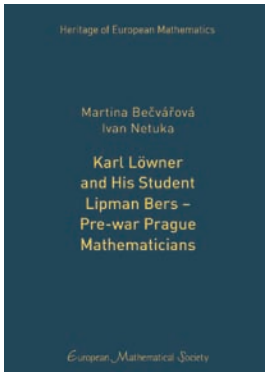
Spectral theory is a diverse area of mathematics that derives its motivations, goals and impetus from several sources. In particular, the spectral theory of the Laplacian on a compact Riemannian manifold is a central object in differential geometry. From a physical point a view, the Laplacian on a compact Riemannian manifold is a fundamental linear operator which describes numerous propagation phenomena: heat propagation, wave propagation, quantum dynamics, etc. Moreover, the spectrum of the Laplacian contains vast information about the geometry of the manifold.
This book gives a self-containded introduction to spectral geometry on compact Riemannian manifolds. Starting with an overview of spectral theory on Hilbert spaces, the book proceeds to a description of the basic notions in Riemannian geometry. Then its makes its way to topics of main interests in spectral geometry. The topics presented include direct and inverse problems. Addressed to students or young researchers, the present book is a first introduction in spectral theory applied to geometry. For readers interested in pursuing the subject further, this book will provide a basis for understanding principles, concepts and developments of spectral geometry.



Della Dumbaugh (University of Richmond, USA) and Joachim Schwermer (University of Vienna, Austria)
Emil Artin and Beyond – Class Field Theory and *L*-Functions (Heritage of European Mathematics)

ISBN 978-3-03719-146-0. 2015. 248 pages. Hardcover. 17 x 24 cm. 68.00 Euro

This book explores the development of number theory, and class field theory in particular, as it passed through the hands of Emil Artin, Claude Chevalley and Robert Langlands in the middle of the twentieth century. Claude Chevalley's presence in Artin's 1931 Hamburg lectures on class field theory serves as the starting point for this volume. From there, it is traced how class field theory advanced in the 1930s and how Artin's contributions influenced other mathematicians at the time and in subsequent years. Given the difficult political climate and his forced emigration as it were, the question of how Artin created a life in America within the existing institutional framework, and especially of how he continued his education of and close connection with graduate students, is considered. In particular, Artin's collaboration in algebraic number theory with George Whaples and his student Margaret Matchett's thesis work "On the zeta-function for ideles" in the 1940s are investigated. A (first) study of the influence of Artin on present day work on a non-abelian class field theory finishes the book. The volume consists of individual essays by the authors and two contributors, James Cogdell and Robert Langlands, and contains relevant archival material.



Martina Bečvářová (Czech Technical University and Charles University, Prague, Czech Republic) and Ivan Netuka (Charles University, Prague, Czech Republic)
Karl Löwner and His Student Lipman Bers – Pre-war Prague Mathematicians (Heritage of European Mathematics)

ISBN 978-3-03719-144-6. 2015. 304 pages. Hardcover. 17 x 24 cm. 78.00 Euro

This monograph is devoted to two distinguished mathematicians, Karel Löwner (1893–1968) and Lipman Bers (1914–1993), whose lives are dramatically interlinked with key historical events of the 20th century. K. Löwner, Professor of Mathematics at the German University in Prague (Czechoslovakia), was dismissed from his position because he was a Jew, and emigrated to the USA in 1939. Earlier, he had published several outstanding papers in complex analysis and a masterpiece on matrix functions. In particular, his ground-breaking parametric method in geometric function theory from 1923, which led to Löwner's celebrated differential equation, brought him world-wide fame and turned out to be a cornerstone in de Branges' proof of the Bieberbach conjecture. L. Bers was the final Prague Ph.D. student of K. Löwner. His dissertation on potential theory (1938), completed shortly before his emigration and long thought to be irretrievably lost, was found in 2006. It is here made accessible for the first time, with an extensive commentary, to the mathematical community.
This monograph presents an in-depth account of the lives of both mathematicians, with special emphasis on the pre-war period. Each of his publications is accompanied by an extensive commentary, tracing the origin and motivation of the problem studied, and describing the state-of-art at the time of the corresponding mathematical field.

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