Abstract. Introduced by Bökstedt-Hsiang-Madsen in the nineties, topological cyclic homology is a manifestation of the dual visions of Connes-Tsygan and Waldhausen to extend de Rham cohomology to a noncommutative setting and to replace algebra by higher algebra. The cohomology theory that ensues receives a denominator-free Chern character from algebraic $K$-theory, used by Hesselholt-Madsen to evaluate the $p$-adic $K$-theory of $p$-adic fields. More recently, Bhatt-Morrow-Scholze have defined a “motivic” filtration of topological cyclic homology and its variants, the filtration quotients of which give rise to their denominator-free $p$-adic Hodge theory $\mathcal{A}$.\[120x344]Mathematics Subject Classification (2010): 19D55, 14F30, 13D03, 19D50.\]

Introduction by the Organisers

Cyclic homology was introduced by Connes and Tsygan in the early eighties to serve as an extension of de Rham cohomology to a noncommutative setting. The negative version of cyclic homology receives a trace map from algebraic $K$-theory, which extends the classical Chern character and roughly records traces of powers of matrices. This trace map is a powerful rational invariant of algebraic $K$-theory. Indeed, a theorem of Goodwillie from 1986 shows that, rationally, the discrepancy for $K$-theory to be invariant under nilpotent extensions of rings agrees with that for negative cyclic homology; and a theorem of Cortiñas from 2006 shows similarly that, rationally, the discrepancy for $K$-theory to preserve cartesian squares of rings agrees with that for negative cyclic homology.

In the early seventies, Boardman and Vogt planted the seeds for the higher algebra that was only fully developed much later by Joyal and Lurie, and, later in the decade, Waldhausen extended Quillen’s definition of algebraic $K$-theory from
the rings of algebra to the (connective $E_1$-)rings of higher algebra. Waldhausen advocated that the initial ring $S$ of higher algebra be viewed as an object of arithmetic and that the cyclic homology of Connes and Tsygan be developed with the ring $S$ as its base. In his philosophy, such a theory should be meaningful integrally as opposed to rationally.

In 1985, Bökstedt realized Waldhausen’s vision as far as Hochschild homology is concerned, and he named this new theory topological Hochschild homology. (A similar construction had been considered by Breen ten years earlier.) He also made the fundamental calculation that, as a graded ring,

$$\text{THH}_*(\mathbb{F}_p) = \text{HH}_*(\mathbb{F}_p/S) = \mathbb{F}_p[x]$$

is a polynomial algebra on a generator $x$ in degree two. By comparison,

$$\text{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x \rangle$$

is the corresponding divided power algebra, and hence, Bökstedt’s theorem supports Waldhausen’s vision that passing from the base $\mathbb{Z}$ to the base $S$ eliminates denominators. In fact, the base-change map $\text{HH}_*(\mathbb{F}_p/S) \to \text{HH}_*(\mathbb{F}_p/\mathbb{Z})$ can be identified with the edge homomorphism of a spectral sequence

$$E^2_{i,j} = \text{HH}_i(\mathbb{F}_p/\pi_*(S)) \Rightarrow \text{HH}_{i+j}(\mathbb{F}_p/S),$$

so apparently the higher stable homotopy groups of spheres, which Serre had proved to be finite, are exactly the right size to eliminate the denominators in the divided power algebra.

The appropriate definition of cyclic homology relative to $S$ was given in 1993 by Bökstedt-Hsiang-Madsen. It involves a new ingredient that is not present in the Connes-Tsygan cyclic theory, namely, a Frobenius. The nature of this Frobenius is now much better understood thanks to the work of Nikolaus-Scholze [18], and we will use this work as our basic reference. As in the Connes-Tsygan theory, the circle group $\mathbb{T}$ acts on topological Hochschild homology, and by analogy, we may define negative topological cyclic homology and periodic topological cyclic homology to be the homotopy fixed points and the Tate construction of this action, respectively:

$$\text{TC}^-(A) = \text{THH}(A)^{\mathbb{T}} \quad \text{and} \quad \text{TP}(A) = \text{THH}(A)^{\mathbb{T}}.$$

There is always a canonical map from homotopy fixed points to the Tate construction, but, after $p$-completion, the Frobenius gives rise to another such map and the Bökstedt-Hsiang-Madsen topological cyclic homology is the homotopy equalizer of these two maps:

$$\text{TC}(A) \xrightarrow{\varphi_p} \text{TC}^-(A) \xrightarrow{\text{can}} \text{TP}(A).$$

Topological cyclic homology receives a trace map from algebraic $K$-theory, which is called the cyclotomic trace map, and Dundas-McCarthy-Goodwillie showed that the discrepancy for $K$-theory to be invariant under nilpotent extensions agrees integrally with that for topological cyclic homology. Similarly, by work of Geisser-Hesselholt and Dundas-Kittang, the discrepancy for $K$-theory to preserve cartesian squares of rings agrees integrally with that for topological cyclic homology.
Calculations of algebraic $K$-groups, or rather the homotopy groups of the $p$-adic completion of the $K$-theory spectrum, by means of the cyclotomic trace begin with the calculation that said trace map

$$K(\mathbb{F}_p) \to \text{TC}(\mathbb{F}_p)$$

induces an isomorphism of $p$-adic homotopy groups in non-negative degrees. The Dundas-McCarthy-Goodwillie theorem together with continuity results of Suslin and Hesselholt-Madsen then show that the same is true for

$$K(\mathbb{Z}_p) \to \text{TC}(\mathbb{Z}_p)$$

and, more generally, for finite algebras over the ring of Witt vectors in a perfect field of characteristic $p$. This was used by Hesselholt-Madsen in 2003 to verify the Lichtenbaum-Quillen conjecture for $p$-adic fields, by evaluating the relevant topological cyclic homology, and one of the goals of the Arbeitsgemeinschaft was to understand this calculation.

The theories $\text{TC}^\ast$ and $\text{TP}$ are of significant independent interest, since they are closely related to interesting $p$-adic cohomology theories, both new and old. The precise relationship was established only recently by work of Bhatt-Morrow-Scholze that defines “motivic filtrations” on THH and related theories, the graded pieces of which are $p$-adic cohomology theories such as crystalline cohomology and the $A\Omega$-theory of [2]. For example, if $X$ is a scheme smooth over a perfect field of characteristic $p$, then the $j$th graded pieces of $\text{TC}$, $\text{TC}^\ast$, and $\text{TP}$ form a homotopy equalizer

$$
\begin{array}{ccc}
\mathbb{Z}_p(j) & \xrightarrow{\text{Fil}^j} & \text{Fil}^j \Gamma_{\text{crys}}(X/W(k)) \\
 & \mapright{\cong} & R\Gamma_{\text{crys}}(X/W(k)).
\end{array}
$$

A second goal of the Arbeitsgemeinschaft was to understand these filtrations.

Since the questions that we consider are mainly in the $p$-complete setting and for $E_\infty$-algebras (in fact, usual commutative rings!), we largely restrict our attention to this case, and in particular work with $p$-typical cyclotomic spectra.

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