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Computability Theory

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Abstract. Computability and computable enumerability are two of the fundamental notions of mathematics. Interest in effectiveness is already apparent in the famous Hilbert problems, in particular the second and tenth, and in early 20th century work of Dehn, initiating the study of word problems in group theory. The last decade has seen both completely new subareas develop as well as remarkable growth in two-way interactions between classical computability theory and areas of applications. There is also a great deal of work on algorithmic randomness, reverse mathematics, computable analysis, and in computable structure theory/computable model theory.

The goal of this workshop is to bring together researchers representing different aspects of computability theory to discuss recent advances, and to stimulate future work.

Mathematics Subject Classification (2010): 03Dxx; Secondary: 03C57, 68Qxx.

Introduction by the Organisers

Computability theory is one of the main branches of mathematical logic. It explores the computational limitations of mathematics. At the center of this area are notions of relative computation and various induced hierarchies. Classical concepts include the degrees of unsolvability, the arithmetical and analytic hierarchies, and many other methods of calibrating relative complexity. Principal applications have been to algorithmic randomness (e.g., Kolmogorov complexity), mathematical logic, algebra, analysis, and proof theory (such as reverse mathematics).

A number of deep tools have been developed in the area. These include priority methods, effective forcing methods, and sophisticated coding techniques. While there have been some memorable recent results clarifying the pure theory, much of current research is devoted to using these techniques to distill the effective content of and give insight into applications. This activity has given rise to several new international conferences: *Computability and Complexity in Analysis (CCA)* since 1995, *Computability, Complexity and Randomness (CCR)* since 2004, and *Computability in Europe (CiE)* since 2005, which all emphasize particular applications; the CiE meetings also include physicists, biologists, and linguists.

Moreover, there have been several recent major advances, such as the solution of a 50 year-old question on the definability of the total degrees (M. Cai, Ganchev, Lempp, J. Miller, M. Soskova), solutions to longstanding questions on tilings and entropy (Hochman, Meyerovich, 2010), and proofs that torsion-free abelian groups cannot have useful invariants (Downey, Montalbán, 2008).

Here are some of the strands we seek to draw together at this meeting:

Fundamentals. The basic notion of the area is that of a "reduction": $A \leq B$ means that A "is computable" from B. The traditional notion is Turing reducibility, introduced by Turing in 1939. Turing reducibility yields the Turing degrees, used to measure the complexity of unsolvable problems. There has been a great deal of work on the structure of the Turing degrees and its restriction to the computably enumerable degrees. Post had asked whether all noncomputable c.e sets have the same Turing degree. The negative solution by Friedberg and Muchnik in the late 1950's introduced the "priority method", a signature method in the subject. A great deal is known about the Turing degrees and the c.e. degrees, but some fundamental problems remain open, in particular whether there is a nontrivial automorphism, and the related "Bi-Interpretability Conjecture".

There are other degree structures, for example, the *enumeration degrees*. The past few years have seen a great deal of progress in understanding the enumeration degrees and their connection with the Turing degrees. Much of the progress is due to M. Soskova and her collaborators. This work culminated in a result stating that the Turing degrees sit inside the enumeration degrees as a definable subset (M. Cai, Ganchev, Lempp, J. Miller, M. Soskova, 2016).

From these fundamentals we derive various hierarchies which align themselves with *logical definability*. For example, Σ^0_1 means that a problem is essentially some kind of countable computable unbounded search, like the famous Halting Problem; and being Σ^0_1-hard means being as complicated as any other Σ^0_1-problem and corresponds to classical problems like Hilbert's 10th problem. Similarly, Σ^1_1 correlates to a search through all $2^{\aleph_0}$ many functions from $\mathbb{N}$ to $\mathbb{N}$, so being Σ^1_1-hard means that this search cannot be simplified. Isomorphism between countable structures is naturally Σ^1_1, and hence, when it is shown that an isomorphism problem for a class of structures is Σ^1_1-hard, then there cannot be any simplifying invariants, like dimension. Downey and Montalbán used this method to prove that torsion-free abelian groups cannot have any useful classifying invariants.

Inspired by related work in algorithmic randomness, significant portions of recent work have focused on computable approximations of noncomputable objects via “tracing” and limit approximations. For example, the sets that can only compute functions f where we can give a computable set of possibilities as the value for $f(n)$ turn out to be precisely the sets that cannot derandomize a certain class of random reals. A major recent unifying program for approximations was set out by Cholak, Downey and Greenberg.

Algorithmic Randomness. A natural arena for computability theory is the area of algorithmic randomness. This area tries to give meaning to randomness for individual sequences and strings. Typical questions are: When is a real more random than another, what is the computational power of a random real, or sets of random strings, how can we understand “almost everywhere” behavior in mathematics? The hierarchies associated with algorithmic randomness and those of computability theory interrelate. A remarkable example of this stems from work on “K-trivial” sets. This analysis has led to new results on the structure of the c.e. sets, “natural” solutions to Post’s problem, new randomness notions (Bienvenu, Greenberg, Kučera, Nies, Turetsky, 2016) and entirely new algorithmic methods.

Analysis and randomness. The early interactions between computability and randomness have developed into widespread applications in computable analysis, ergodic theory, subshifts of finite type, tiling problems and even number theory. Randomness and genericity align themselves to differentiability of effective functions (Brattka, J. Miller, Nies).

Randomness is tied to effective dimension ergodic theory starting with V’yugin’s proof (1997) that Martin-Löf randomness suffices for the effective Birkhoff theorem. Work in symbolic dynamics shows close relationships between entropy, effective and classical Hausdorff dimension, and Kolmogorov complexity (Simpson, 2015). A standard notion in randomness is the halting probability, and this has been found to be quite natural and to turn up in places apparently removed from such considerations, e.g., in the classification of dimensions of subshifts of finite type (Hochman, Meyerovich, 2010), and the complexity of Julia Sets (Braverman, Yampolsky, 2006). Classical computable analysis remains a very active field of research, and these new interactions with randomness are invigorating both areas.

Computable model theory. In computable model theory, we consider structures with effective presentations. Typically, we look at the interplay of definability and algorithmic behavior. For example, a sufficiently decidable structure is computably categorical iff it can be “named” by an infinitary computable formula. Many of the original limitations were established by unnatural “pathological” examples, and much recent work seeks to answer what “tame”, or “natural”, behavior is. The *degree spectrum* of a relation R on a structure $\mathcal{A}$ is the set of Turing degrees of images of R in computable copies of $\mathcal{A}$. There are examples of computable structures with a relation whose degree spectrum is strange. Harrison-Trainor has investigated spectra of relations “on a cone” and showed a number of dichotomies on spectra. Csima and Harrison-Trainor considered degrees of categoricity on a cone. Again, there are examples with strange degrees of categoricity, but on a cone, there is tame behavior, the degree of categoricity is Δ^0_α for some α.

A recurrent idea is that nice model-theoretic properties of the theory should make it easier to understand the complexity of the models. There has been major progress recently, bounding the complexity of countable models of strongly minimal theories (Andrews, Knight), and bounding the possibilities on which countable models of strongly minimal theories are computable (Andrews, Lempp).

Of course, this work is related to computable algebra, where we deal with concrete algebraic structures like groups, rings and fields. Hirschfeldt, Khoussainov, Slinko, and Shore in 2002 gave general conditions on a class of structures that permit effective coding and decoding of graphs. Among the original examples of classes that satisfy the conditions are partial orderings, lattices, rings, integral domains, commutative semigroups and 2-step nilpotent groups. Now, fields have been added fields to this list (R. Miller, Schoutens, 2012; R. Miller, Poonen, Schoutens, Shlapentokh). For the fine structure of *particular* algebraic objects, new methods and complex computability techniques seem to be needed. Recent results include an analysis of the complexity of radicals in rings by Conidis, and the isomorphism problem for completely decomposable groups by Downey and Melnikov. A new line of research is to work in uncountable structures.

Reverse mathematics and proof theory. Reverse mathematics seeks to classify mathematical results according to the proof-theoretical resources needed for them. The techniques of this area and computability theory are close, and there is much cross-fertilization. Initially, the program of reverse mathematics found many important mathematical theorems equivalent to just one of five systems, linearly ordered. More recent work, especially on principles related to combinatorics, has produced a large number of new systems. A new result in this area was announced at the 2012 Oberwolfach meeting by Chong, Slaman and Y. Yang, who gave an amazing proof separating certain variations of Ramsey's Theorem using non-standard models. This technique is certain to yield further results.

Related to this are pre-orderings suggested by Kolmogorov on "problems"; Medvedev and Muchnik, respectively, made this intuition precise as strong and weak reducibilities on "mass problems", i.e., subsets of Baire space or Cantor space. Among the mass problems, we may consider those asking for a copy of a given structure, or certain collections of random sets. In recent years a number of new reducibilities, e.g. Weihrauch reducibility and computable reducibility, emerged that generalize the Medvedev and Muchnik reducibilities in the sense that parameterized "problems" are considered. These reducibilities allow a more resource sensitive and uniform version of reverse mathematics that can be directly approached with computability-theoretic techniques. So far, problems from computable analysis (Brattka, Gherardi, 2011) and combinatorial problems (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2016) have been classified in this approach.

The workshop is organized by Vasco Brattka, Rodney G. Downey, Julia F. Knight, and Steffen Lempp, and includes 28 talks and two open problem sessions. Some of the open problems discussed are listed in the last section of this report. The slides for the talks can be found on Steffen Lempp's website at `http://www.math.wisc.edu/~lempp/conf/OW18/OW18slides.htm`.

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