Mathematisches Forschungsinstitut Oberwolfach

Report No. 10/2017

DOI: 10.4171/OWR/2017/10

Mini-Workshop: Perspectives in High-Dimensional Probability and Convexity

Organised by
Joscha Prochno, Hull
Christoph Thäle, Bochum
Elisabeth M. Werner, Cleveland

5 February – 11 February 2017

Abstract. Understanding the geometric structure of systems involving a huge amount of parameters is a central problem in mathematics and applied sciences today. Here, geometric and analytical ideas meet in a non-trivial way and powerful probabilistic tools play a key role in many discoveries. Two essentially independent areas of mathematics concerned with high-dimensional problems are asymptotic geometric analysis and information-based complexity. In this Mini-Workshop we brought together researchers from both fields to explore the connections and form synergies to develop new perspectives.

Mathematics Subject Classification (2010): 52A20 60D05 65D30 (11K38 52A22 52A23 65C10).

Introduction by the Organisers

The mini-workshop: *Perspectives in high-dimensional probability and convexity* was organized by Joscha Prochno (Hull, UK), Christoph Thäle (Bochum, Germany) and Elisabeth M. Werner (Cleveland, USA). In total, 16 participants attended the workshop.

High-dimensional systems are frequent in mathematics and applied sciences, and understanding of high-dimensional phenomena has become an increasingly important topic. Mathematical disciplines that are most strongly related to such phenomena are functional analysis, convex geometry and probability theory. In fact, a new area emerged, now called asymptotic geometric analysis, which is at the very core of the crossroads of these disciplines and bears connection to mathematical physics and theoretical computer science as well. The last two decades have seen a tremendous growth in this area. Far reaching results were obtained and various powerful techniques, mainly of a probabilistic flavor, have been developed. A major stimulus and impulse for the theory is the famous hyperplane conjecture which stays unsolved till the present day. Against this background, we explored new perspectives during this workshop and brought together three different groups of researchers working on problems involving high-dimensional set-ups, and thus contributing different angles on high-dimensional phenomena.

To enable participants with different background to form synergies, we kept the number of talks at a minimum and restricted ourselves to 6 survey talks, each ending with a set of possible open problems for group work. Three of the lectures were given on Monday and three on Tuesday. The speakers were:

- Olivier Guédon (Paris):
  Perspectives on the Kannan-Lovász-Simonovits conjecture
- Aicke Hinrichs (Linz):
  Discrepancy and dispersion of point distributions
- Carsten Schütt (Kiel):
  Random polytopes and approximation
- David Alonso-Gutiérrez (Zaragoza):
  Random convex sets verifying the hyperplane and variance conjectures
- Jan Vybiral (Prague):
  IBC: Approximation problems and lower bounds
- Matthias Reitzner (Osnabrück):
  High-dimensional random polytopes

Monday and Tuesday afternoon ended with an open problem session. In those two sessions, the already mentioned potential problems from the survey talks were discussed in more detail and new problems were added. After agreeing on a final set of problems, the participants selected a particular question of their interest and we split up in essentially two groups. For the remainder of the week, the groups worked on and discussed these respective problems (a detailed description of the problems and the outcome of the group discussion is given below). This resulted in additional short talks in the smaller working groups to clarify certain aspects of the theory or to present some important results related to the question. On Thursday morning, each of the groups gave a 15 minutes talk, addressing the progress they had made, stating results and/or presenting the (technical) problems they had run into as well as the different approaches they had tried. On Friday before lunch, the groups gathered for a final update on their work. These presentations also formed the closure of the mini-workshop.

One special focus in this workshop was on the connections between asymptotic geometric analysis and information-based complexity (IBC), two young and essentially independent areas dealing with high-dimensional problems. A group of participants from both areas decided to work on questions in this direction. At the center of attention was the problem of the minimal dispersion of point sets. This is a classical problem in computational geometry, which is related to the notion of discrepancy and the approximation of rank-one tensors. The goal is to find the largest empty axis-parallel box amidst a point set $\mathcal{P}_n = \{t_1, \dots, t_n\}$ inside the unit cube $[0,1]^d$ (in high dimensions). If we denote by $\mathcal{B}_{\text{ax}}$ the set of axis-parallel boxes inside $[0,1]^d$, then the minimal dispersion of this set is defined to be

$$\operatorname{disp}_{\mathcal{B}_{\text{ax}}}(n,d) = \inf_{\substack{\mathcal{P}_n \subseteq [0,1]^d \\ |\mathcal{P}_n| = n}} \sup_{\substack{B \in \mathcal{B}_{\text{ax}} \\ B \cap \mathcal{P}_n = \emptyset}} \operatorname{vol}_d(B)\,.$$

There are two very recent notable results providing upper and lower bounds in this setting and, as part of the group work and discussions, two members prepared short talks of 30 minutes to explain their proofs to the other participants. In fact, the two bounds are of a different order in d and n and the gap is quite huge. For instance, the lower bound is logarithmic in d, but is expected to display a linear behavior in d (in fact, this linear term can be seen in the upper bound up to a logarithmic factor). The participants of the workshop who focused on that particular problem are still working on improving both best known lower and upper bound. For more details we refer to the group work report below.

Another aspect that has been addressed during the week was the central limit problem for the volume of random simplices in high dimensions. For $1 \le r \le d$ let $X_0, X_1, \dots, X_r$ be independent random points that are uniformly distributed in the normalized d-dimensional cube $[-\sqrt{3}, \sqrt{3}]^d$ and denote by V_r the r-volume of their convex hull, which is almost surely a simplex of dimension r. It is known from the literature that for fixed r, the random variables V_r satisfy a central limit theorem, as $d \to \infty$. One of the open problems in this area is to show asymptotic normality for V_r also in the high-dimensional regime, where $r = r(d) \to \infty$. During the workshop we considered the particularly attractive and extremal full-dimensional case $r = d$. While in this situation there is no central limit theorem for V_d itself, it turns out that the random variables $\log V_d$ are asymptotically Gaussian. To show this new central limit theorem, the participants working on this problem have split up into further subgroups to work on the details of the proof. In particular, after having connected the question to already existing result in random matrix theory (more precisely, random determinants), it has become necessary to understand certain details in the literature. These were afterwards presented in short talks in order to put together all pieces for the proof. Currently, the participants from this working group are writing down their result and try to extend it in different directions. We expect that this will eventually lead to a joint publication.
More details on this aspect of the mini-workshop will be explained in the group work report below.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Mini-Workshop: Perspectives in High-Dimensional Probability and Convexity

Table of Contents