Mini-Workshop: Surreal Numbers, Surreal Analysis, Hahn Fields and Derivations

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Abstract. New striking analogies between H. Hahn’s fields of generalised series with real coefficients, G. H. Hardy’s field of germs of real valued functions, and J. H. Conway’s field No of surreal numbers, have been lately discovered and exploited. The aim of the workshop was to bring quickly together experts and young researchers, to articulate and investigate current key questions and conjectures regarding these fields, and to explore emerging applications of this recent discovery.

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Introduction by the Organisers

The field No of surreal numbers is a real closed field which simultaneously contains the real numbers and the ordinal numbers. Since its discovery, it has ever been the object of intense research. It was established by various authors (e.g. Alling, Costin, v. d. Dries, Ehrlich, Gonshor, Kruskal, to cite a few) that No is a universal domain for real algebra (in the sense that every real closed field whose domain is a set can be embedded in No), that it admits an exponential function, and an interpretation of the restricted real analytic functions, making it into a model of the elementary theory of the field of real numbers endowed with the exponential function and all the restricted real analytic functions.

In the last decade, it has been conjectured that No is a universal domain for real differential algebra. To this end, an immediate goal is to equip No with a derivation compatible with the exponential and with its natural structure as a
Hahn field. Moreover such a derivation should formally behave like the natural derivation on the germs at infinity of univariate real valued functions belonging to a Hardy field. A related conjecture is that $\mathbb{N}o$ can also be viewed as a universal domain for generalized series fields equipped with an exponential function, such as the logarithmic-exponential series of v. d. Dries, Macintyre and Marker, the exponential-logarithmic series of F.- V. Kuhlmann and S. Kuhlmann and the field of transseries of v. d. Hoeven. This lead to the explicit conjecture by S. Kuhlmann and Matusinski (2011 & 2012) that $\mathbb{N}o$ is a field of exponential-logarithmic transseries and can be equipped with a Hardy-type series derivation. Progress in this direction was achieved by Berarducci and Mantova (2015 & 2016), showing that the surreal numbers have a natural transseries structure and admit a compatible Hardy-type derivation.

The purpose of this mini-workshop was to bring together quickly a small team of senior and junior mathematicians, who felt the time was ripe to mine the theory of surreal numbers for insights into topics such as real algebra, asymptotic analysis, model theory, set theory and the foundations of mathematics. We concentrated our efforts mainly in three research directions: (A) study derivation, integration and composition operators on those fields, (B) investigate their algebraic and arithmetic properties, (C) explore their applications to computable analysis. The structure of the meeting was organised around these goals. Leading experts gave keynote lectures (tutorials/surveys) on the first and second day, in order to promptly introduce the background, and bring in the cutting-edge research. The special session organised by Mantova and Matusinski was held already on the afternoon of the second day, thereby intensifying the impact. The last day was dedicated to the talks of the doctoral students, who in the meanwhile had time to absorb the lively presentation and discussion style, and make it into their own. We now describe in detail the course of events.

1. Special lectures (tutorials) by distinguished experts

A four parts introductory tutorial on the field of surreal numbers (Matusinski), surreal exponentiation (Mantova), initial embeddings in the surreal, in particular the work of Ehrlich on that topic (Fornasiero) and generalised series as germs of surreal functions (Berarducci) occupied most of the day on Monday. This was followed on Tuesday morning by a two-part introductory tutorial on transseries (Point) and $\kappa$-bounded series (S. Kuhlmann). Wednesday morning was dedicated to further survey talks on quasi-analytic classes (Speissegger), closed ordered differential fields (Tressl) and integer parts of real closed fields (S. Kuhlmann).

2. Special session on derivations induced by right shifts

One of the key steps in the recent paper by Berarducci and Mantova in order to construct a derivation on $\mathbb{N}o$, consists in identifying the class of log-atomic monomials and defining suitable derivatives of these numbers, hence addressing the above conjecture raised by S. Kuhlmann and Matusinski. The latter authors (2011) gave a criterion relating pre-logarithms and derivations for formal series, via
the action of a right-shift automorphism on the chain of fundamental monomials. The aim of the session was to clarify the connection between the two approaches. This special session lasted half of a day. The first hour and a half were dedicated to two informal talks about right-shift automorphisms and surreal derivations. A two hour problem session followed (see paragraph on open problems below).

3. Research talks and talks of doctoral students

L’Innocente and Mantova reported in two consecutive talks on their recent joint work related to the arithmetic and algebraic properties of Oz. Berarducci presented the work of Costin-Ehrlich-Friedman (who did not attend the meeting) on integration, and Kaiser presented a slides talk on Lebesgue measure theory over the surreals. All the graduate students gave exciting talks. Galeotti explained the interest of surreals in descriptive set theory and computable analysis. Lehericy presented his work on asymptotic couples, and Krapp on o-minimal exponential fields. Kaplan lectured on his joint work with Ehrlich and Müller on quasi-ordered fields.

4. Open problems and questions:

Question 1. The field $\mathbb{T}$ of logarithmic exponential series has a natural derivation $\partial$ and an exponential map $\exp$. It is known that $(\mathbb{T},\exp)$ is an elementary extension of $(\mathbb{R},\exp)$ [36] and $(\mathbb{T},\partial)$ is an elementary substructure of $(\mathbb{No},\partial)$ with the derivation of [10].

1. Is $(\mathbb{T},\partial,\exp) \preccurlyeq (\mathbb{No},\partial,\exp)$?
2. Every Hardy field can be embedded in $(\mathbb{No},\partial)$ as a differential field [8]. Now, let $(K,\partial,\exp)$ be a Hardy field closed under $\exp$. Can we embed it in $(\mathbb{No},\partial,\exp)$?

Question 2. Describe the field operations of $\mathbb{No}$ using the sign sequence representation.

Question 3. Let $i = \sqrt{-1}$. Is there a good way to introduce $\sin$ and $\cos$ on $\mathbb{No}$ and an exponential map on $\mathbb{No}[i]$? Is there a surreal version of the $p$-adic numbers?

Question 4. The field $\mathbb{R}(\omega)$ of omega-series is the smallest subfield of the surreal numbers containing $\mathbb{R}(\omega)$ and closed under log, exp and sums of arbitrary summable sequences. In [11] it is shown that this field has a unique natural derivation and composition operator and contains, as differential subfields, the various variants of transseries fields (LE and EL-series). Each $f \in \mathbb{R}(\omega)$ (hence in particular any transseries) determines a function $\hat{f} : \mathbb{No}^\mathbb{R} \to \mathbb{No}$ on positive infinite surreal numbers.

1. Does the structure $(\mathbb{No},+\times,\exp,\hat{f})_{f\in\mathbb{R}(\omega)}$ have good model theoretic properties? It may be conjectured that, restricting the various $f$ to some infinite half-line $(a, +\infty)$, one obtains an o-minimal structure. A preliminary question is whether the intermediate value theorem holds for $\hat{f}$.
(2) Can we find a good composition operator on the whole of \( \mathbb{No} \)?

(3) Are the EL-series elementary equivalent to \( \mathbb{R} \langle \langle \omega \rangle \rangle \) as a differential field?

**Question 5.** Let \( \kappa \) be an uncountable cardinal such that \( \kappa^{<\kappa} = \kappa \) and \( \mathbb{R}_\kappa \) be the Cauchy completion of \( \mathbb{No}_\kappa \). As shown in [86], one can code elements of \( \mathbb{R}_\kappa \) by binary sequences of length \( \kappa \) and define a notion of computability over \( \mathbb{R}_\kappa \) using Turing machines running for \( \kappa \) many steps. Let \( f : \mathbb{R}_\kappa \to \mathbb{R}_\kappa \) be \( \kappa \)-computable (hence continuous). Does \( f \) satisfy the intermediate value theorem?

**Question 6.** Consider a field of \( \kappa \)-bounded generalized power series \( \mathbb{R}((G))_\kappa \) as in [77]. Is there a Kaplansky differential embedding theorem for \( \mathbb{R}((G))_\kappa \)?

**Question 7.** Let \( \kappa_x \in \mathbb{No} \) be the \( \kappa \)-number indexed by \( x \in \mathbb{No} \) and let \( \lambda_x \in \mathbb{No} \) be the log-atomic number indexed by \( x \in \mathbb{No} \). For every ordinal \( \alpha \) we know that \( \lambda_{\omega\alpha} = \kappa_\alpha \) (see Cor. 2.10 in [8]). What is the function \( f \) such that \( \lambda_{f(x)} = \kappa_x \) for every \( x \in \mathbb{No} \)?

**Question 8.** The surreal numbers \( \mathbb{No} \) contain a largest exponential subfield \( \mathbb{R}((L)) \) satisfying axiom ELT4 of [75]. Are there distinct surreal derivations on \( \mathbb{No} \) with the same restriction to \( \mathbb{R}((L)) \)? See [10] for background.

**Question 9.** Let \( \kappa > \omega \) be a regular cardinal. It is known that the Hahn fields \( \mathbb{R}((G)) \) do not admit an exponential map, but for suitable \( G \) the \( \kappa \)-bounded subfields \( \mathbb{R}((G))_\kappa \) do admit an exp [77]. Despite the fact that \( \mathbb{No} \) is sometimes loosely described as a Hahn field \( \mathbb{R}((G)) \), the correct analogy is rather with the \( \kappa \)-bounded version \( \mathbb{R}((G))_\kappa \). A general question is to explore these analogies and find ways of introducing derivations on \( \kappa \)-bounded series compatible with an exponential function. In particular one would like to find \( \partial \) and exp such that \( (\mathbb{R}((G))_\kappa, \text{exp}, \partial) \) is isomorphic to a fragment of \( \mathbb{No} \) with some surreal derivation (see [10]). One such fragment could be \( \mathbb{No}_\kappa \), the subfield of \( \mathbb{No} \) consist of the surreal numbers of length \( < \kappa \). Another suitable fragment could be the intersection of \( \mathbb{No}_\kappa \) with the field \( \mathbb{R}((L)) \) defined in [10]) (the largest subfield of \( \mathbb{No} \) satisfying axiom ELT4 of [75]).

**Question 10.** Given a real closed field \( K \) admitting an integer part which is a model of \( \text{PA} \), does \( K \) admit a total exponential function? Kuhlmann proved that it admits a left-exponential function.

**Question 11.** Is every exponential group the value group of an exponential field? See the abstract of S. Kuhlmann on Integer parts.

**Question 12.** Does every RCF admit a normal integer part? See the abstract of S. Kuhlmann on Integer Parts.

**Question 13.** Every real closed field \( F \) admits a truncation closed embedding into a field of Hahn series and a corresponding truncation closed integer part \( Z \subseteq F \) (which is a model of Open Induction). Is the class of all such truncation closed integer parts an elementary class? See the abstract of S. Kuhlmann on Integer Parts.
Question 14. In the work of L’Innocente and Mantova (see their abstracts), irreducibility in rings of generalized power series is studied with the help of a degree function deg. Can we extend deg to a field valuation?

Question 15. Can one obtain an integration theory for semialgebraic differential forms on semialgebraic \( \mathbb{N}_o \)-submanifolds, including Stokes theorem? Can one extend the measure and integration theory on \( \mathbb{N}_o \) beyond the semialgebraic and globally subanalytic category? See the abstract of T. Kaiser for references.

Question 16. Can one characterise the subset \( \mathbb{Q} \) of \( \mathbb{N}_o \) in terms of sign-sequences?

Question 17. Can one extend the simplicity order of \( \mathbb{N}_o \) to functions? In which sense \( + \) is the simplest function increasing in both arguments? Is \( \exp \) the simplest homomorphism from \( (\mathbb{N}_o, +) \) to \( (\mathbb{N}_o^{\infty}, \times) \) such that for all \( n \in \mathbb{N} \) and positive infinite \( x \in \mathbb{N}_o \) we have \( \exp(x) > x^n \)?

Question 18. Can one describe an integer part of \( \mathbb{N}_o \) which is a model of true arithmetic? The existence of such an integer part should follow by the saturation properties of \( \mathbb{N}_o \), but can one construct such an integer part explicitly? (without the axiom of choice, say).

5. Conclusion and outlook

In the opinion of the organizers, this mini-workshop was a great success, because it achieved exactly the desired impact on the subject. We think that the following facts were fundamental to this very special event:

(1) one third of the participants were doctoral students working in the general area of research, and eager to learn the background quickly and collect research problems,

(2) we organized 20 talks as well as an afternoon dedicated to a special discussion and problems session,

(3) since we had 20 talks distributed among 15 participants on the one hand all the graduate students were given opportunity to speak about their ongoing research, on the other hand we could dedicate more than one talk to some topics, thus allowing in-depth treatment,

(4) we highly recommended black-board talks and encouraged audience to ask questions, speakers to include open problems, conjectures, or simply challenging exercises in their talks,

(5) the special session was an opportunity to draw a research road map for the upcoming year on the topic of the mini-workshop.

All this produced a tremendous synergy; participants junior and senior were discussing intensely during the breaks and well into the evenings. Doctoral students collected new ideas and inspirations for their dissertations. Several collaborations were initiated during this short meeting. Our reporter Lorenzo Galeotti invested a lot of effort in keeping track and recording the open questions and exercises (see paragraph above). We intend to apply for a follow-up mini-workshop at MFO very soon, to keep the acquired research momentum and collaborations.
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