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## Asymptotic Phenomena in Local Algebra and Singularity Theory

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ABSTRACT. The goal of this workshop was to highlight, and further, the interactions between local algebra and singularity theory. The timing was serendipitous for both subjects have witnessed tremendous progress recently, much of which was reported at the workshop.

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### Introduction by the Organisers

In the recent past there have been a number of spectacular developments in local algebra and singularity theory, the subject of this Oberwolfach workshop. Three outstanding conjectures have been settled (in the past six months) and there has been significant progress on two others; what is more, at Oberwolfach we got news that there has been a major breakthrough on another long-standing open problem. Fortunately, all the researchers responsible for these developments had been invited to the workshop, and most were able to participate and present their work. All this made for a lively and memorable gathering. There were more than fifty participants, from across the world; twenty three lectures, each an hour long, were scheduled. Here are some of the highlights.

*Hochster's direct summand conjecture.* The direct summand conjecture asserts that regular local rings are direct summands of their module-finite extensions. It is one in a network of homological conjectures in local algebra, formulated by Hochster, that have generated a tremendous amount of activity in the last 50 years. They had largely been resolved for commutative rings that contain a field

or in low dimension. The direct summand conjecture (which has been known to be equivalent to various other homological conjectures, such as the canonical element conjecture and the improved new intersection theorem, among others) remained open in mixed characteristic until this past summer when André announced a proof of it based on Scholze's theory of perfectoid spaces. Bhatt delivered the opening lecture of the workshop, outlining a simplified proof of this conjecture, and an extension to the geometric setting; this again makes critical use of perfectoid spaces. By popular demand, Bhatt gave a second, more informal, lecture on Tuesday evening, giving an overview of perfectoid theory. It is clear that perfectoid theory is going to have an enormous impact on commutative algebra.

*Eisenbud-Goto regularity conjecture.* Over thirty years ago, Eisenbud and Goto conjectured a bound for the Castelnuovo-Mumford regularity of a prime ideal in terms of its multiplicity. This bound was proved by Gruson, Lazarsfeld, and Peskine for curves, by Pinkham and Lazarsfeld for smooth complex surfaces, and for some smooth 3-folds by Ran. Furthermore in dimensions three and four, Kwak proved regularity bounds that are only slightly worse than the one suggested by the conjecture. McCullough and Peeva (who were both at the workshop, as was Eisenbud) found a counterexample this summer, using an ingenious construction that yields, moreover, examples where the regularity is not bounded by any polynomial function of the multiplicity. McCullough gave a wonderful talk on these, and more recent, developments on this topic.

*Resolution of singularities.* In the early 1960s, Hironaka proved the existence of a resolution of singularities of a reduced algebraic scheme over a field of characteristic zero. This result has had spectacular application in many areas of mathematics. Because of the lack of hypersurfaces of maximal contact in positive characteristic, this problem has remained open over fields of positive characteristic, not to mention mixed characteristic. Abhyankar proved resolution of singularities for 3-folds of characteristic larger than five in the mid 1960s and Lipman proved resolution for reduced excellent surfaces in the mid 1970s. Recently, resolution for reduced excellent schemes of dimension three has been proven by Cossart and Piltant. This is the most general theorem which can be true in dimension three. Schober, who works with Cossart and Piltant, gave a talk in the workshop on this result, explaining the result and giving an outline of the proof.

*Lech's multiplicity conjecture.* Fifty years ago Lech conjectured that if  $R \rightarrow S$  is a flat local ring extension, the multiplicities satisfy  $e(R) \leq e(S)$ ; he proved it when  $\dim R \leq 2$ . Ma spoke about his recent work establishing an amazing inequality when  $R$  contains a field; this settles the case  $\dim R = 3$ . His proof uses Hilbert-Kunz multiplicities and local Chern characters to tackle rings of positive characteristic; he deduces the result for rings containing a field of characteristic zero using Artin approximation and reduction to positive characteristic.

*Stillman's projective dimension conjecture.* About ten years ago, Stillman conjectured that there is an upper bound on the projective dimension of an ideal in

a polynomial ring, in terms of the number of generators of the ideal and their degrees; the remarkable point is that the bound is independent of the dimension of the ring. A few months ago Ananyan and Hochster posted a preprint on the arXiv that settles this conjecture. Regrettably, neither of them were present at this workshop. We were fortunate that Caviglia agreed to give an overview of the proof, which is a spectacular tour de force, involving a subtle study of subalgebras generated by regular sequences, and a complicated induction scheme.

*Buchsbaum-Eisenbud-Horrocks conjecture on Betti numbers.* This conjecture, from the 1970s, states that the  $i$ th Betti numbers of a module of finite length over a regular local ring of dimension  $d$  is at least  $\binom{d}{i}$ ; in particular, the total Betti number is at least  $2^d$ . It is related to the problem of finding minimal ranks of syzygies of vector bundles on the punctured spectrum of the ring. It can also be viewed as the local algebra analogue of Halperin's conjecture that if a real torus of dimension  $d$  acts almost freely on a finite CW complex  $X$ , then the total rank of the rational homology of torus is at least  $2^d$ . At the workshop Walker announced that he had settled the conjecture concerning total Betti numbers just a few days ago! He gave a beautiful talk, with a complete proof of his result that covers also the case of modules of finite projective dimension over general local rings. The key new idea comes from K-theory, mainly the use of Adams operations on perfect complexes. It is clear that this opens up a whole new arsenal of techniques for use in local algebra, with a promise of further progress in this direction.

In addition to these topics, the talks in the workshop covered a range of topics of current interest in local algebra and singularity theory.

Presentations were given on recent progress towards resolution of singularities in positive characteristic and higher dimension. Teissier discussed his proof of local uniformisation along a maximal rank valuation in all dimensions and in positive characteristic by a close analysis of the associated graded ring of a local ring with respect to a valuation. Hauser gave an example showing that one of the main invariants used in resolution of singularities can have unexpected pathological behavior in higher dimension and positive characteristic. Villamayor discussed a systematic way to improve invariants of resolution under permissible blow ups in a ramified map to a nonsingular scheme of positive characteristic.

Singularities in positive characteristic were also the focus of the talks of Dao and Tucker. The latter discussed his recent result that the étale local fundamental group of an F-regular scheme (which can be viewed as the positive characteristic analog of a Kawamata log terminal singularity) at a geometric point is finite. Dao talked about the asymptotic behavior of the local cohomology modules of thickenings (that is to say, powers, or Frobenius powers) of an ideal in a local ring.

The structure of free resolutions of modules over local rings continues to be an active area of research, and many of the talks in the workshop were related to this topic. Şega's talk focused on the question of the rationality of Poincaré series of modules over compressed local rings. Berkesch explained some special features of homological algebra in multigraded settings. Avramov, Eisenbud, and Peeva

spoke about (invariants of) minimal free resolutions of modules over complete intersection rings. On the more representation theoretic side of things, Buchweitz presented a complete description of the graded maximal Cohen-Macaulay modules over graded one dimensional Gorenstein rings.

The interaction between commutative algebra, algebraic geometry and combinatorics has a long tradition dating back, at least, to the 1970's with the pioneering work of Stanley and Hochster. Four talks at the meeting can be broadly considered as part of this area: Murai's talk on the properties of double h-polynomial of Buchsbaum complexes, Varbaro's presentation on a surprising connection between the Castelnuovo-Mumford regularity of monomial ideals and the virtual cohomological dimension of hyperbolic Coxeter groups, Huh's proof of the "top-heavy" conjecture of Dowling and Wilson for certain matroids employing geometric methods, and Srinivasan's analysis of homological properties of certain monomial curves.

Finally, two talks were devoted to introduce the audience to new research directions. In Erman's talk the problem of detecting Noether normalizations over finite fields was discussed and a solution based on a variant of Poonen's closed point sieve was presented. Römer presented results concerning Hilbert series of algebraic objects that have a finite description up to the action of an infinite group as part of the recent theory of "finite up-to symmetry" ideals which is connected with stable asymptotic behaviour of group representations.

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