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The Renormalization Group

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Abstract. The renormalization group was originally introduced as a multiscale approach to quantum field theory and the theory of critical phenomena, explaining in particular the universality observed e.g. in critical exponents. Since then it has become a hugely important tool in statistical mechanics, condensed matter and high energy physics. More recently, renormalization has also played a decisive role in mathematics as a method of proof, applicable in quantum field theory, differential equations, probability, and other fields. The workshop has focused on new developments along the lines of these two traditions. Besides discussing methodical progress and current applications, we have explored new challenges and problems that may in the future be tackled with the help of the renormalization group.

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Introduction by the Organisers

The renormalization group has its origins in quantum field theory and statistical mechanics. The idea is to implement scale transformations, possibly combined with other operations such as averaging, on function spaces and operator algebras, and hence on the action functionals and Hamiltonians that define classical or quantum ensembles and dynamics on these spaces. In this way, the transition from a microscopic to a macroscopic scale in systems with many degrees of freedom is formulated in terms of a dynamical system. In the past half-century, this concept has led to a uniform description of diverse physical phenomena, providing a natural explanation why physical phenomena exhibit universal features. In the sequel, the method has entered mathematics as a method of proof that has gradually been applied to solve a great variety of problems in mathematical physics, but also in probability and partial differential equations. To provide a context, we first sketch the roots and some important developments of the field. (We do not attempt completeness, as this would require too much space.)

Renormalization in quantum field theory (QFT) started out in the 1950s as perturbative renormalization, with the aim of getting well-defined (finite) results at any fixed order in an expansion in a coupling parameter, usually organized in terms of Feynman graphs. It was converted to a mathematical theory in the 1960s and early 1970s, notably by Bogoliubov, Parasiuk, Hepp, and Zimmermann, culminating in an inductive proof of finiteness to all orders of the Green functions of physically important QFTs like quantum electrodynamics. Zimmermann's famous forest formula provides an explicit solution to the recursion of perturbative renormalization, which captures both the analytical and the combinatorial aspects of the procedure in a concise way. In parallel, Epstein and Glaser developed the viewpoint of renormalization as the definition of causal products of distributions. The group of scaling transformations plays a role because the careful mathematical analysis requires introducing a renormalization scale, the variation of which leads to the renormalization group. In the 1970s, 't Hooft and Veltman proved renormalizability of nonabelian gauge theory to first order in a loop expansion, using dimensional renormalization. Breitenlohner and Maison then established the action principle and the BRST identities for this scheme to all orders, and Piguet and Sibold treated supersymmetric theories in the 1980s.

Renormalization in statistical mechanics originated in the idea that at a critical point of a statistical mechanical system, scale invariance should set in. Kadanoff first introduced block-spin transformations in the 1960s. Wilson continued and extended these ideas and developed the renormalization group as a combination of averaging operations and scale transformations. Wilson's RG phenomenology of a flow of effective actions parametrized by length scale shifted the focus from studying individual fixed models to considering a dynamical system on "the space of all theories". This new formulation of renormalization changed the concepts and techniques used in theoretical physics completely, far beyond their original application to critical phenomena. It also allowed to go beyond formal perturbation expansions and to construct models of QFT and statistical mechanics in a mathematical sense.

The renormalization group in the sense of Kadanoff and Wilson was developed into a mathematical theory in the 1980s in pioneering work by Gallavotti and Nicolò, by Gawędzki and Kupiainen, by Feldman, Magnen, Rivasseau and Sénéor, and by Bałaban. Beginning with these works, the method has led to breakthroughs in mathematical results about quantum field theories and models of statistical mechanics. Indeed, virtually all mathematical constructions of quantum field theories in dimensions three or higher involve the RG, and the RG has become a powerful method in the study of systems with infinitely many degrees of freedom in general, and in the construction and analysis of non-Gaussian measures on infinite-dimensional spaces.

These two seemingly different strands of renormalization were joined in the 1970s and 1980s in the work of the above-mentioned people, and by Wegner, Polchinski, and others. In particular, Polchinski used Wilson's flow equation RG to give a proof of perturbative renormalizability that bypasses most of the combinatorial complications of earlier works, and which has since been developed into a powerful tool that has led to many new results, e.g. to a recent proof of perturbative renormalizability of the standard model of elementary particle physics on curved spacetime. Brydges and Kennedy used Polchinski's formulation to shed new light on the forest formula, and to develop new tree expansions that have become standard tools in constructive QFT. Brydges and Yau developed a setup in which the decomposition in large and small fields is avoided.

Much further work has also been done in simplifying perturbative renormalization, extending it to more general situations, and providing new mathematical viewpoints. The conceptually very clear method of Epstein and Glaser was revived by Scharf and collaborators, and then further developed using tools of microlocal analysis by Radzikowski, and then by Brunetti and Fredenhagen and coworkers, and by Hollands and Wald. Kreimer, and then Connes and Kreimer, provided a Hopf-algebraic formulation of the graph operations implementing renormalization in formal perturbation theory. Costello gave another variant of Wilson's effective-action approach to perturbative renormalization.

Last, but not least, starting with the work of Feldman and Trubowitz, as well as Benfatto and Gallavotti, in the early 1990s, the renormalization group has led to a host of new results in mathematical condensed-matter theory, in particular laying rigorous foundations for Luttinger and Fermi liquid theory and numerous new results about spin systems. Moreover, it has placed the proof of Bose-Einstein condensation in the thermodynamic limit within reach.

The 2016 Oberwolfach workshop *The Renormalization Group*, the fourth of its kind at Oberwolfach, organised by M. Disertori (Bonn), M. Salmhofer (Heidelberg) and and W. De Roeck (Leuven), was attended by 43 participants, from universities and research institutes mainly in Europe and North America, with backgrounds ranging from theoretical physics over mathematical physics to pure mathematics.

The talks and discussions covered most of the broad range of topics exposed above. There were talks on novel views and methods of perturbative renormalization (Nguyen, Rejzner), on the operator product expansion (Hollands), on the rigorous construction and analysis of models in classical and quantum statistical mechanics and condensed-matter physics (Feldman, Lohmann, Giuliani, Porta, Pizzo), on probabilistic applications (Brydges, Slade), on specific analysis tools (Buchholz), on applications to nontrivial QFT models motivated by high-energy physics and combinatorics (Abdesselam, Rivasseau), and spectral theory and quantum dynamics (Sigal). Moreover, there were talks about topics that may become interesting for renormalizers (Fröhlich, Gawedzki, Warzel).

The presentations of T. Nguyen and K. Rejzner concerned the Batalin-Vilkovisky approach to the quantization and renormalization of theories with large symmetry groups (e.g. theories with local gauge or diffeomorphism invariance). Rejzner used the causal perturbation theory approach of Epstein and Glaser in Lorentzian signature, while Nguyen used Costello's version of the effective action approach on Riemannian spaces, with a standard heat kernel regularization, to derive results about the absence of anomalies in nonlinear sigma models.

S. Hollands presented his work on the operator product expansion (OPE), touching a number of different aspects, namely the convergence of the expansion in Euclidian scalar field theory (in every fixed order in perturbation theory), proven by an extension of Polchinski's flow equation method together with Holland and Kopper, the idea of regarding the OPE as the defining feature of a QFT, and a novel construction of the Gross-Neveu model via a new recursion formula for the OPE.

V. Rivasseau talked about renormalization of tensor field theory, which is related to models of quantum gravity, and which poses interesting new features and aspects of renormalization theory.

S. Buchholz, one of the PhD researchers at the workshop, outlined a novel approach to finite-range decompositions of covariances, an analytical tool that has recently helped to simplify cluster expansions needed in renormalization. Among other things, the finite-range decomposition allows to treat non-analytic perturbations, which occur in applications to models for elasticity.

Several talks dealt with applications of the mathematical renormalization group. We had two talks, by J. Feldman and M. Lohmann, on the program of controlling interacting bosons with spontaneous symmetry breaking. This program has as its medium-term goal a proof of Bose-Einstein condensation in the thermodynamic limit, which can be considered one of the most important currently open problems in mathematical physics. G. Slade spoke about $O(n)$ models with a long-range kinetic term, engineered so that these models are slightly below the upper critical dimension. By applying machinery from earlier work with Brydges and Bauerschmidt, the critical point can be studied. I.M. Sigal described some new results on open quantum systems consisting of baths coupled to small systems. In particular, decoherence rates have been exhibited and contrasted with the rate of thermalization in such models. This program relies on the use of the Feshbach map spectral renormalization group. The talk by M. Porta dealt with the universality of the Hall conductance. This universality is very well understood for non-interacting electrons where it is a consequence of topology. Porta and coworkers employed fermionic perturbation theory to extend this to interacting electrons, as well. A. Giuliani presented an application of fermionic perturbation theory to planar dimer models. These models are solvable in the close-packed limit, when the interaction only includes a hard-core repulsion that prevents dimers from overlapping. The new results allow to treat more general dimer interactions. This gives rise to a continuous family of critical models, with varying critical exponents. A. Abdesselam considered field theory on the p-adic numbers, which can be viewed as a model in which the hierarchical approximation to the renormalization group becomes exact. In particular, one of the achievements in this framework is the construction of a theory with anomalous power-law decay of correlations. The talk by A. Pizzo concerned the ground state of interacting bosons approaching the mean field limit. A lot of work has been done on this problems in the past decade. Pizzo's approach is based on the Feshbach renormalization group and it yields new results on that ground state, making precise the well-known Bogoliubov approximation.

Finally, we had a few interesting talks that were not, at least not explicitly, connected to the renormalization group: D. Brydges outlined an application of the lace expansion to the $n = 2$ component ϕ^4 theory in high dimensions, proving that critical exponents are given by mean field theory. S. Warzel reported on work on planar Ising models. While it has been known already since Onsager that 2-dimensional models often can be analyzed in terms of free fermions, the presented work has significantly extended and generalized this connection. R. Kotecký presented work on the transition from a metastable supercooled gas phase to the stable liquid phase in the Widom-Rowlinson model. The talk of K. Gawedzki focused on periodically driven non-interacting systems, thus bringing us to non-equilibrium physics. Such periodically driven systems can be described by a Floquet operator, which replaces the Bloch Hamiltonian. The topic addressed here was how to define topological invariants for such systems, like, e.g. for topological insulators.

J. Fröhlich gave an overview of the 'Gauge Theory of States of Matter', developed by him and coworkers throughout the 90's. This is a program that classifies relevant effective field theories in condensed matter, thus predicting important phenomena including topological insulators. The talk also touched on more recent works, proposing models for dark matter and dark energy.

The schedule was set up so as to leave enough time for individual discussions and collaboration. The feedback from the participants was positive, suggesting to us that the meeting has been fruitful. The Oberwolfach atmosphere and the perfect organization at the Forschungsinstitut, as well as the friendly and efficient service by the staff were greatly appreciated.

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