Mathematisches Forschungsinstitut Oberwolfach

Report No. 20/2016

DOI: 10.4171/OWR/2016/20

Arbeitsgemeinschaft: The Geometric Langlands Conjecture

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3 April – 9 April 2016

Abstract. The Langlands program is a vast, loosely connected, collection of theorems and conjectures. At quite different ends, there is the geometric Langlands program, which deals with perverse sheaves on the stack of G-bundles on a smooth projective curve, and the local Langlands program over p-adic fields, which deals with the representation theory of p-adic groups. Recently, inspired by applications to p-adic Hodge theory, Fargues and Fontaine have associated with any p-adic field an object that behaves like a smooth projective curve. Fargues then suggested that one can interpret the geometric Langlands conjecture on this curve, to give a new approach towards the local Langlands program over p-adic fields.

Mathematics Subject Classification (2010): 11S37, 11F70, 22E50, 14H60, 14D24, 22E57.

Introduction by the Organisers

The Arbeitsgemeinschaft *The Langlands program: From global unramified geometric to local ramified arithmetic*, organised by Laurent Fargues, Dennis Gaitsgory, Peter Scholze and Kari Vilonen, brought together over 50 students and experts working in different aspects of the Langlands program, algebraic geometry, p-adic Hodge theory and related areas, with a diverse geographic and mathematical background.

The goal of the workshop was to understand the statement of Fargues' conjecture, which builds a bridge between the geometric Langlands conjectures, usually stated in the global and unramified setting, with the more classical 'arithmetic' Langlands conjectures, specifically the (ramified) local Langlands conjecture over

p-adic fields. Thus, the program was split (roughly) in half, with some of the lectures giving an overview of the statement and proof (for GL_2) of the geometric Langlands conjecture, and the other half leading up to the formulation of Fargues' conjecture, including various necessary background talks on perfectoid spaces and the Fargues–Fontaine curve. Because the program was quite dense, various extra discussion sessions were scheduled in the afternoon. This led to an extremely intense and fruitful exchange between researchers from these different areas.

Let us give a brief introduction to these questions. The Langlands program emerged as an organizational principle in the theory of automorphic forms. Classically, automorphic forms are (roughly) functions on symmetric domains G/K where G is a real Lie group and $K \subset G$ a maximal compact subgroup, which are required to be invariant under the action of an arithmetic subgroup $\Gamma \subset G$. The prototypical example is the case of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-space, giving rise to modular forms and Maaßforms. On the space of automorphic forms, one has a large space of symmetries, classically given by differential operators, and Hecke operators. This big space of operators on automorphic forms allows one to extract spectral data. One of the Langlands conjectures predicts that this same spectral data is also seen in (apparently unrelated) arithmetic situations. The prototypical example is the relation between rational modular forms of weight 2 and elliptic curves E over $\mathbb{Q}$, which relates Hecke eigenvalues with the number of $\mathbb{F}_p$-rational points of E.

In the modern formulation, one starts with a reductive group G over $\mathbb{Q}$, and one regards $\mathbb{Q}$ as the function field of the "compact curve" $\overline{\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$. For each place v of this curve, i.e., v is either a prime number p or the archimedean place ∞, one has the completion $\mathbb{Q}_v$ of $\mathbb{Q}$ at v, which are either the p-adic numbers, or the reals $\mathbb{R}$. One can also form the adèles $\mathbb{A}$ of $\mathbb{Q}$, which is the subring of $\prod_v \mathbb{Q}_v$ given by the condition that almost all components are integral.

An automorphic representation of G is (roughly) an irreducible representation of $G(\mathbb{A})$ that occurs in the space of L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Any irreducible representation π of $G(\mathbb{A})$ decomposes as a (restricted) tensor product

$$\pi = \bigotimes_v \pi_v$$

of irreducible representations π_v of $G(\mathbb{Q}_v)$. The rough statement of the local Langlands conjecture says that for each v, the datum of π_v is equivalent to a representation of the absolute Galois group of $\mathbb{Q}_v$, with values in the Langlands dual group.[1] The rough statement of the global Langlands conjecture is that if π is automorphic with a suitable condition on π_∞, then there is a representation of the absolute Galois group of $\mathbb{Q}$, inducing all these representations of the local absolute Galois groups. Moreover, one should be able to go in the converse direction.

A completely parallel conjecture can be formulated for the function field F of a projective smooth curve over a finite field, in place of $\mathbb{Q}$. Several simplifications

[1]At least at $v = \infty$, one has to use the Weil group of $\mathbb{R}$.

occur in this case, the most important being that the space $G(F)\backslash G(\mathbb{A}_F)$ is 0-dimensional, so most analytic aspects of the problem are gone. Notably, many of Langlands' conjectures have been proved in this case by Drinfeld, L. Lafforgue and V. Lafforgue.

The (global, unramified) geometric Langlands program

The geometric Langlands program emerged as a geometric way of looking at Langlands' conjectures in the case of a function field. It is most directly related to the classical picture when looking at the global, everywhere unramified correspondence.

Let C be a smooth projective curve over any field k, and let us continue to denote by F its function field. For any closed point x of C, we write $\mathcal{O}_x$ for the completion of the structure sheaf at x, and F_x for its quotient field. Let $\mathbb{A}_F = (\prod_x \mathcal{O}_x) \otimes F$ be the adèles. If k is a finite field, then everywhere unramified automorphic representations correspond to functions on the double quotient

$$G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x)\ .$$

The basic observation is that if Bun_G denotes the stack of G-bundles on C, then there is a bijection

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbb{A}_F)/G(\prod_x \mathcal{O}_x)\ .$$

If k is a finite field, then functions on $\mathrm{Bun}_G(k)$ can be geometrized by perverse sheaves on Bun_G: Any perverse sheaf gives a function of k-points by looking at traces of Frobenius on the stalks. The analogue of the Hecke action is given by the action of Hecke correspondences on the stack of G-bundles.

Looking at the other side of the correspondence, everywhere unramified Galois representations are precisely local systems on C (with values in the L-group ${}^LG$ of G). Thus, the geometric Langlands conjecture predicts that for every ${}^LG$-local system E on C, there is a perverse sheaf $\mathrm{Aut}_{\mathrm{E}}$ on Bun_G which satisfies a suitable Hecke equivariance property. For $G = \mathrm{GL}_n$, it has been proved by Frenkel, Gaitsgory and Vilonen, following earlier work of Drinfeld, and Laumon.

If k is a finite field, this conjecture implies the global unramified classical Langlands conjecture by passing to the corresponding function on $\mathrm{Bun}_G(k)$.

However, when trying to generalize to ramified representations, it is very difficult to see the arithmetic of supercuspidal representations of $G(\mathbb{F}_p((t)))$, and its relation with irreducible Galois representations of the absolute Galois group of $\mathbb{F}_p((t))$ in this picture. The basic reason is that the geometric picture is automatically compatible with extensions of the base field $k = \mathbb{F}_p$, whereas these arithmetic phenomena are not.

Fargues' conjecture

At his MSRI lecture in December 2014, Fargues stated a most striking conjecture. In recent work with Fontaine, for any non-archimedean local field K (i.e., K is a finite extension of $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$), he had constructed a certain scheme X_K

over K, which behaves like a smooth projective curve over an algebraically closed field, but is not of finite type. This construction was motivated by considerations in p-adic Hodge theory.

Fargues' observation was that if one interprets the global unramified geometric Langlands conjecture on this curve, one ends up with a statement that encodes most conjectural properties of the local ramified arithmetic Langlands conjecture over K. One critical difference is that the automorphism group of the trivial G-torsor is not the algebraic group G, but the locally profinite group $G(K)$, so (perverse) sheaves on the stack of G-bundles naturally give rise to representations of $G(K)$. One can hope that this makes it possible to adapt methods from the geometric Langlands program to make progress on the local Langlands conjectures.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

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