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Mini-Workshop: Mathematical Foundations of Isogeometric Analysis

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Abstract. Isogeometric Analysis (IgA) is a new paradigm which is designed to merge two so far disjoint disciplines, namely, numerical simulations for partial differential equations (PDEs) and applied geometry. Initiated by the pioneering 2005 paper of one of us organizers (Hughes), this new concept bridges the gap between classical finite element methods and computer aided design concepts.

Traditional approaches are based on modeling complex geometries by computer aided design tools which then need to be converted to a computational mesh to allow for simulations of PDEs. This process has for decades presented a severe bottleneck in performing efficient simulations. For example, for complex fluid dynamics applications, the modeling of the surface and the mesh generation may take several weeks while the PDE simulations require only a few hours.

On the other hand, simulation methods which exactly represent geometric shapes in terms of the basis functions employed for the numerical simulations bridge the gap and allow from the beginning to eliminate geometry errors. This is accomplished by leaving traditional finite element approaches behind and employing instead more general basis functions such as B-Splines and Non-Uniform Rational B-Splines (NURBS) for the PDE simulations as well. The combined concept of Isogeometric Analysis (IgA) allows for improved convergence and smoothness properties of the PDE solutions and dramatically faster overall simulations.

In the last few years, this new paradigm has revolutionized the engineering communities and triggered an enormous amount of simulations and publications mainly in this field. However, there are several profound theoretical issues which have not been well understood and which are currently investigated by researchers in Numerical Analysis, Approximation Theory and Applied Geometry.

Mathematics Subject Classification (2010): 65xx, 41xx, 51xx.

Introduction by the Organisers

Isogeometric Analysis (IgA) is a new paradigm which has been mainly established in the engineering sciences over the past eleven years. It merges two so far disjoint disciplines, namely, numerical simulations for partial differential equations (PDEs) and applied geometry. This new concept was initiated by the pioneering 2005 paper of one of us organizers (Hughes) and bridges the gap between classical finite element methods and computer aided geometric design concepts.

Traditional approaches to solve PDEs on complicated domains are based on modeling complex geometries by computer aided geometric design tools. These need to be converted to a computational mesh to allow for simulations of PDEs. This process has for many decades presented a severe bottleneck in performing efficient simulations, even though computers have become more and more powerful. For example, for applications involving complex fluid dynamics, the modeling of the surface and the mesh generation ("by hand") may take several weeks while the PDE simulations require only a few hours.

The main idea of Isogeometric Analysis which overcomes this bottleneck is the following. For exactly representing geometric shapes, one typically employs piecewise polynomials or rational functions as basis functions. If the same functions are used for the numerical simulations of the PDEs, one eliminates geometry errors. The new paradigm consists of leaving traditional finite element approaches behind and employing instead more general basis functions such as B-Splines and Non-Uniform Rational B-Splines (NURBS) for the PDE simulations as well. The combined concept of Isogeometric Analysis (IgA) allows for improved convergence and smoothness properties of the PDE solutions and dramatically faster overall simulations.

In the last few years, this new paradigm has revolutionized the engineering communities and triggered an enormous amount of simulations and publications mainly in this field. However, there are several profound theoretical issues which have not been well understood and which are currently investigated by researchers in Numerical Analysis, Approximation Theory and Applied Geometry. These problems firstly concern multiscale techniques for variational problems discretized by B-splines and NURBS, namely,

- multilevel solvers;
- hierarchical spaces, adaptivity;
- construction of non-tensor product functions;
- error estimation, convergence and complexity estimates;
- quadrature.

For example, the issue of constructing optimal preconditioners independent of the polynomial degree of the basis functions is except for involved auxiliary space methods a hot topic. A-posteriori error estimation and the possibility to develop adaptive methods with respect to both the mesh and the polynomial degree is not mathematically understood in the IgA framework. A bottleneck for computations for variational formulations of PDEs even on uniform grids is currently the set-up of linear systems of equations which require highly efficient quadrature rules for B-splines and NURBS.

A second thematic focus of the Mini-Workshop addressed the recent revival of collocation methods, which provide significant advantages by employing higher order B-Splines and NURBS in combination with an IgA framework. These methods have been popular some decades ago in the numerical PDE community but have then been essentially abandoned, due to a largely missing mathematical foundation with respect to, e.g., error estimates. They are still used in applications, e.g., in boundary element methods. Collocation methods are based on point evaluations of the PDE in strong form. In principle, this is a dramatic advantage over variational formulations which require efficient quadrature rules. Collocation schemes also allow for quick evaluations of nonlinearities. These advantages have recently triggered new and promising work in all fields of computational mechanics.

So far, an error analysis for collocation methods exists only for PDEs on one-dimensional domains or under very high smoothness assumptions on the PDE solution. For higher dimensions, the issue how to select the collocation points to derive corresponding estimates is not understood. This entails that also a convergence and complexity theory is not yet available.

The goal of the Mini–Workshop *Mathematical Foundations of Isogeometric Analysis* organised by Thomas J.R. Hughes (Austin), Bert Jüttler (Linz), Angela Kunoth (Köln) and Bernd Simeon (Kaiserslautern) was to bring together some leading scientists from IgA and the mathematically relevant fields. We wanted to start with brainstorming in an atmosphere of a small workshop with not too many participants who are a nice blend of researchers with various backgrounds. The Mini-Workshop was well attended with 17 participants with broad geographic representation.

The participants were experts who are strong in approximation theory (Lyche, Oswald), numerical analysis and multiscale methods (Demlow, Kunoth, Langer, Mantzaflaris, Sangalli, Simeon), applied geometry and geometric design (Harbrecht, Jüttler, Manni, Mourrain, Peters) together with researchers in the engineering sciences with a strong mathematical background in modeling and numerics (Evans, Hughes, Reali). In addition, a young Bachelor student (Akpinar) presented promising first results for approximations of high-dimensional integrals.

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