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Lattices and Applications in Number Theory

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Abstract. This is a report on the workshop on Lattices and Applications in Number Theory held in Oberwolfach, from January 17 to January 23, 2016. The workshop brought together people working in various areas related to the field: classical geometry of numbers, packings, Diophantine approximation, Arakelov geometry, cohomology of arithmetic groups, algebraic modular forms and Hecke operators, algebraic topology. The meeting consisted of a few long talks which included an introductory part to each of the topics in the previous list, and a series of shorter talks mainly devoted to recent developments. The present report contains extended abstracts of all presentations.

Mathematics Subject Classification (2010): 11F75, 11H31, 11H55, 14G40, 20H10, 22E57.

Introduction by the Organisers

The theory of lattices in Euclidean spaces is a very old subject which is still of great interest because of its various connections to other mathematical theories. As is well-known, the "Geometry of Numbers" developed at the end of the nineteenth century by Hermann Minkowski, following the pioneering works of Hermite, had a profound influence on the development of algebraic number theory. One goal of the workshop "Lattices and Applications in Number Theory" was to attest the vitality of this trend in modern number theory and to show how the theory of Euclidean lattices still provide tools for important discoveries. It was also an occasion for researchers from fairly different areas to exchange their ideas.

The meeting brought together 54 mathematicians from 11 countries. There were eleven one-hour talks, aimed at introducing to non-experts the various topics addressed during the week and presenting recent developments as well. Besides, eight shorter talks (45 minutes) on some recent developments were proposed, and two afternoon sessions of short presentations (5×20 minutes) were organised.

We briefly describe below some of the main topics that were addressed during the workshop, and point out some of the results exposed.

- *Arakelov geometry* is the natural setting for a modern view on the "Geometry of Numbers", in which Euclidean lattices appear as a particular instance of Hermitian bundles over arithmetic curves. A striking illustration is the very deep theorem of Zhang on successive minima of Hermitian bundles which can be seen as an analogue of Minkowski's classical theorem about the successive minima of Euclidean lattices. The talks by Gaël Rémond, Éric Gaudron, Jean–Benoît Bost, and Christophe Soulé pertained to this topic. A closed formula for an absolute version (*i.e.* over $\overline{\mathbb{Q}}$) of the Hermite constant was presented in Rémond's talk.
- *Applications of Voronoi's algorithm to arithmetic groups*: In a famous paper dating from 1907, Voronoi defined a face-to-face tiling of the cone of positive semidefinite quadratic forms by so-called *perfect domains* and described an algorithm to enumerate these domains up to $\mathrm{SL}_n(\mathbb{Z})$ equivalence. It was observed in the 1970s (Ash, Soulé) that Voronoi's tesselation could also be used to compute the (co)homology of $\mathrm{SL}_n(\mathbb{Z})$. Since then, and up to very recently, this observation, extended to more general arithmetic groups, gave rise to a substantial amount of work by various researchers. Philippe Elbaz–Vincent explained one of these recent developments in his talk during the meeting (triviality of $K_8(\mathbb{Z})$ and application to the Vandiver Conjecture). The talks by Dan Yasaki and Joachim Schwermer dealt with related topics. Finally, Roland Bacher explained a construction of families of integral perfect lattices of minimum 4 which as a by-product shows that the number of perfect lattices grows at least exponentially in the dimension.
- In the tradition of Siegel's works, *classical modular forms* play a central role in many questions involving lattices (representation numbers, mass formulas, classification of genera). This also was illustrated during the workshop, e.g. in Rainer Schulze–Pillot's and Jeremy Rouse's talks.
- *Arithmetic groups and algebraic modular forms*: A recent development in the study of arithmetic groups is the theory of algebraic modular forms, initially developed by Benedict Gross, where a connection between modular forms theory and Bruhat-Tits buildings of algebraic groups is studied. The intermediate objects again are lattices, which generalise naturally to integral forms of algebraic groups. Also other notions like genera and mass formulas have been transferred to more general arithmetic groups (dating back to works by Borel, Harish–Chandra and Kneser, and more recently in the fundamental works by Gopal Prasad). A comprehensive introduction to the subject was provided in the talk by Joshua Lansky and David Pollack. Other talks pertaining to this topic were those of Jessica Fintzen and Sebastian Schönnenbeck.

Other crucial topics, not falling within the previous categories, were also addressed: packing and energy minimization problems and related applications of semidefinite optimization, counting arguments for dense lattices in certain families, but also algebraic topology and the algebraic theory of quadratic forms.

All speakers were considerate of the great variety of topics related to lattice theory in this conference and addressed their talks to this broad audience. This concept and the open and stimulating atmosphere of the location lead to many discussions. It was also very fruitful for the many young participants.

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