

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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## Reductions of Shimura Varieties

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ABSTRACT. The aim of this workshop was to discuss recent developments in the theory of reductions of Shimura varieties and related topics. The talks presented new methods and results that intertwine a multitude of topics such as geometry and cohomology of moduli spaces of abelian varieties,  $p$ -divisible groups and Drinfeld shtukas,  $p$ -adic Hodge theory, and the Langlands program.

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### Introduction by the Organisers

The workshop *Reduction of Shimura varieties* was attended by 50 participants with broad geographic representation, including a number of young participants. We had 19 talks of 60 minutes each.

Arithmetic properties of Shimura varieties which are encoded in their reduction to positive characteristic are an exciting topic which has roots in classical topics of number theory such as modular forms and modular curves and of algebraic geometry. On the other hand it is a currently very active research field that has contributed to some of the most spectacular developments in number theory and arithmetic geometry in the last twenty years. Shimura varieties are closely related to the Langlands program (classical as well as  $p$ -adic). A particular case is given by moduli spaces of abelian varieties, a classical object of study in algebraic geometry.

The topics of the talks covered the whole subject of reductions of Shimura varieties, and range from the development of new methods to study them, and results on their geometric and cohomological properties to applications both to the

Langlands program and to other parts of number theory and arithmetic geometry. Some talks reported on relations to group theoretic objects and constructions such as affine root systems and affine Grassmannians, or to analogous constructions in equal characteristic such as moduli spaces of shtuka.

### Applications to number theoretic questions

Particular highlights were the talks on applications of the theory.

Wei Zhang explained joint work with Zhiwei Yun which gives expressions for special values of arbitrary derivatives of certain automorphic representations over global fields of positive characteristic in terms of intersection multiplicities on moduli spaces of shtukas. This is a completely new approach: Previously, usually only the leading term coefficient was considered.

Yifeng Liu's talk titled *Bad reduction of Shimura varieties, level raising and Selmer groups* provided an application of the theory of reductions of Shimura varieties to questions with a number-theoretic flavor, more precisely about the vanishing of certain Selmer groups, a geometric level raising theorem, and a reciprocity law for Gross-Schoen cycles.

Fabrizio Andreatta explained, in his talk "Heights of CM points on orthogonal Shimura varieties", his joint work with Eyal Goren, Benjamin Howard and Keerthi Madapusi Pera on the averaged Colmez conjecture about the heights of certain abelian varieties with complex multiplication. This is quite a powerful result: Tsimerman has shown that it gives rise to an unconditional proof of the André-Oort conjecture.

### Shimura varieties and the Langlands program

Several talks explained aspects of the general paradigm that the cohomology of reductions of Shimura varieties should realize local Langlands correspondences — classical and  $p$ -adic. Related aspects such as  $p$ -adic Hodge theory and deformations of Galois representations were also considered.

Imai's and Ivanov's talk concerned the classical local Langlands correspondence: In his talk "Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations", Naoki Imai discussed joint work with Takahiro Tsushima on explicit constructions of part of the local Langlands and Jacquet-Langlands correspondences in the cohomology of the Lubin-Tate perfectoid space. In a similar vein, Alexander Ivanov proposed a construction of the local Langlands correspondence using covers of affine Deligne-Lusztig varieties, much in analogy with classical Deligne-Lusztig theory. The construction works in general; so far it is known that for  $GL_2$  it does realize the local Langlands correspondence.

On the other hand, in his talk *Analytic functions on étale coverings of Drinfeld's upper half-plane*, Gabriel Dospinescu talked about his joint work with Arthur Cesar Le Bras about the geometric realization of Colmez'  $p$ -adic local Langlands correspondence. This proves a conjecture of Breuil showing that Colmez'  $p$ -adic local Langlands correspondence for  $GL_2(\mathbb{Q}_p)$  can be realized using the de Rham complex of higher coverings of Drinfeld upper half plane. They prove that the

”classical” smooth part of the local Langlands is realized in the de Rham cohomology and the  $p$ -adic one is obtained by pulling back a filtration from the cohomology to the de Rham complex.

Xinwen Zhu explained joint work with Ruochuan Liu which shows that all the fibers of a  $\mathbb{Q}_p$  local system over a smooth connected variety over a  $p$ -adic field are de Rham (as a representation of the absolute Galois group of the residue class field) as soon as this is true over a single point. This is an analogue in  $p$ -adic Hodge theory of Deligne’s principle B (which refers to classical Hodge theory). It can be applied to many Shimura varieties since the de Rham property is easy to obtain at special points.

Brandon Levin talked on *Iwahori local models and deformation rings* and discussed interesting applications of the theory of local models of Shimura varieties to the study of deformation rings of Galois representations, in particular to proving instances of the Breuil-Mézard conjecture (specifically, in the case of  $GL_3$  and Hodge-Tate weights  $(2, 1, 0)$ ).

### Geometry of the reduction of Shimura varieties

Several talks concerned the geometric structure of the reductions of Shimura varieties, in particular the natural stratifications on them.

The topic of Yichao Tian’s talk on joint work with David Helm and Liang Xiao was the proof of the Tate conjecture for the special fibers of some unitary Shimura varieties. The cycles one has to construct to this end are found in the supersingular locus; the supersingular locus is a union of Deligne-Lusztig varieties, similarly as in the case studied by Vollaard and Wedhorn and other cases. This is also closely related to the stratifications discussed in Chen’s talk; see below. To show that this produces enough cycles, one then has to study their geometry inside the full special fiber (in particular, their intersection matrix).

In her talk about joint work with Eva Viehmann, Miaofen Chen described a new stratification of affine Deligne-Lusztig varieties which generalizes many of the stratifications which were previously studied.

The question of the non-emptiness of Newton strata, was addressed by Chia-Fu Yu in his talk *Non-emptiness of the basic locus of Shimura varieties*; it also made an appearance in Mark Kisin’s talk, who talked on joint work with Keerthi Madapusi Pera and Sug-Woo Shin about *Honda-Tate theory for Shimura varieties*, proving that in a Shimura variety  $\mathrm{Sh}_K(G, X)$  with  $G_{\mathbb{Q}_p}$  quasi-split, every isogeny class inside the special fiber of a suitable model contains a point which is the reduction of a CM point.

The theme of Benoît Stroh’s talk *Bad reduction and boundary terms* on joint work with Kai-Wen Lan was that the phenomena of bad reduction and of singularities in the boundary (of a toroidal or the minimal compactification) should not interact with each other. This was illustrated by several results, and also played a role in other talks.

Gerd Faltings presented a case of a Shimura variety with bad, but semi-stable reduction — a rare, but very useful situation; his proof relies on an extension of the theory of “filtered modules” that allows to cover the semi-stable case.

Both George Boxer and Jean-Stefan Koskivirta talked on *Generalized Hasse invariants*. Boxer focused on the construction of generalized Hasse invariants in the Siegel case — while the classical Hasse invariant lives on the full moduli space and vanishes precisely on the non-ordinary locus, the generalized Hasse invariants live on the closure of some Ekedahl-Oort stratum and vanish exactly on its boundary. Boxer's results has applications to the construction of Galois representations attached to automorphic representations. Koskivirta reported on joint work with Goldring. The main focus of his talk was how to deal with Shimura varieties of Hodge type; a key tool is the framework of  $G$ -zips.

Michael Rapoport introduced an axiomatic framework for the reduction of general Shimura varieties concerning the existence and the interplay of several stratifications (Newton, Ekedahl-Oort, Kottwitz-Rapoport) which are well-known from the PEL case. In all cases where the axioms are satisfied, several nice consequences follow, such as the non-emptiness of “all” Newton strata that was also adressed above.

Xu Shen presented a generalization of results of Scholze, now allowing to view arbitrary Shimura varieties of abelian type “at infinite level” as perfectoid spaces and to equip them with a Hodge-Tate period map.

Finally, Eike Lau talked on *The image of the crystalline Dieudonné functor*, a very classical and central method, but whose image was up to now still not completely understood.

The unique environment provided by the Mathematisches Forschungsinstitut Oberwolfach stimulated intense discussions and initiated several new cooperations among the participants. All participants immensely enjoyed the workshop and are very grateful for the institute's hospitality.

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