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## Tropical Aspects in Geometry, Topology and Physics

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**ABSTRACT.** The workshop *Tropical Aspects in Geometry, Topology and Physics* was devoted to a wide discussion and exchange of ideas between the leading experts representing various points of view on the subject. The development of tropical geometry is based on deep links between problems in real and complex enumerative geometry, symplectic geometry, quantum fields theory, mirror symmetry, dynamical systems and other research areas. On the other hand, new interesting phenomena discovered in the framework of tropical geometry (like refined tropical enumerative invariants) pose the problem of a conceptual understanding of these phenomena in the “classical” geometry and mathematical physics.

*Mathematics Subject Classification (2010):* Primary 14-xx (algebraic geometry), Secondary 81Txx (quantum field theory, related classical field theories), 53Dxx (symplectic geometry, contact geometry), 37-xx (dynamical systems and ergodic theory).

### Introduction by the Organisers

The workshop *Tropical Aspects in Geometry, Topology and Physics*, organized by Tobias Ekholm (Uppsala), Hannah Markwig (Saarbrücken), Grigory Mikhalkin (Genève), and Eugenii Shustin (Tel Aviv), was held April 26th–May 2nd, 2015. The workshop was well attended by 55 participants from around the world. The program of the workshop consisted of 17 one-hour talks given by leading experts in the subject as well as 4 half hour talks delivered by perspective young researchers. In addition, four informal discussions on open problems and on questions related to the main topics of the workshop were ran during this week. Extended abstracts of the talks and reports on the discussions follow these introductory notes.

The current state of tropical geometry can be characterized by a wide spread of tropical objects, techniques and ideas in various fields of mathematics and physics. For the further development of the topic it becomes vital to understand these interactions deeper and to collect ideas, approaches and problems coming from different sides which are linked to tropical geometry. Thus, the idea of the workshop was to focus on interactions of tropical geometry with algebraic, symplectic, and combinatorial geometry, low-dimensional topology, mathematical physics, on the exchange of ideas coming from different areas as well as the search for new perspective problems and research directions. Two trends come together nowadays: on one side, the development of tropical geometry have been largely motivated and stimulated by its deep links to geometry, topology and physics; on the other hand, new phenomena and techniques discovered within the tropical geometry framework suggest challenging problems of conceptual understanding and elaboration of these results in the “classical” geometry, topology and physics. We shortly comment on these two trends and on how they were reflected in the talks and discussions during the workshop.

Enumeration of real and complex curves by means of tropical tools has been one of the leading directions in tropical geometry from the very beginning. Surprisingly enough the tropical enumerative geometry has brought purely tropical enumerative invariants like real tropical relative and higher genus invariants, broccoli invariants, and, perhaps, the most interesting – refined (quantum) Block-Goettsche invariants. These refined invariants and their “classical” explanation and construction was one of the main topics discussed by the workshop participants, and it was reflected in talks by Y. Soibelman, M. Abouzaid and in informal discussions led by Y. Soibelman and G. Mikhalkin. So, Y. Soibelman has suggested an interpretation of the refined curve count based on the mirror symmetry for the Fukaya category of the total space of a complex integrable system, where non-trivial B-models give rise to quantization of the structure sheaf of the mirror dual. Closely related to this quantization of the family Floer homology, again in the context of mirror symmetry, was the subject of the talk by M. Abouzaid. Another idea to understand the refined curve count came from G. Mikhalkin’s observation that, for the rational curves on toric surfaces having fixed real or purely imaginary intersection points with the toric divisors, properly defined areas of their amoebas attain a discrete set of values, and the refined count of these curves coincides with the corresponding refined Block-Goettsche tropical count. In turn, in the talk by B. Kol arose an idea to regard the real (and potentially refined) curve count as a supersymmetric Witten index.

Real and tropical enumerative geometry and their interaction were addressed in talks by I. Itenberg and J. Rau, and in the discussion led by P. Georgieva. In particular, I. Itenberg and P. Georgieva introduced new real enumerative invariants, relative (Itenberg) and positive genus (Georgieva), which so far have no a reasonable tropical interpretation. In turn, the main message of J. Rau’s talk was a tropical way to compute real double Hurwitz numbers.

Toric geometry and combinatorics of convex (lattice) polyhedra are an imminent part of constructions and techniques in tropical geometry, and several talks addressed these topics: V. Batyrev presented stringy Chern classes of singular toric varieties, S. Galkin linked quantum cohomology of Fano three-folds with toric degenerations of toric three-folds, N. Kalinin estimated the irrationality degree of hypersurfaces in toric varieties via the combinatorics of their Newton polytopes, a nice application of tropical geometry to the generalized Robinson-Schensted-Knuth correspondence was shown by G. Koshevoy, E. Lupercio has exhibited a theory of non-commutative toric varieties, which, in particular, provides a conventional proof of McMullen inequalities for irrational convex polytopes. T. Nishinou jointly with T. Y. Yu developed a version of the patchworking construction, based, in fact, on specific toric degenerations, in order to prove that immersed trivalent tropical curves in tropical abelian surfaces (equivalently, periodic tropical curves in the plane) are tropicalizations of algebraic curves in abelian surfaces.

The non-Archimedean geometry, Berkovich spaces and tropical varieties came together in the talks by W. Gubler and M. Temkin. The non-Archimedean and Berkovich geometry enrich tropical varieties with new structures, while the tropical picture helps to clarify complicated constructions.

An interaction of tropical geometry with symplectic geometry and mirror symmetry is one of the most perspective and promising research directions. These topics have been presented in the talks by leading experts M. Abouzaid, Y.-G. Oh, Y. Soibelman and in the discussion led by L. Katzarkov. Among the young researchers, H. Argüz demonstrated tropical techniques in development of (symplectic) Floer theory for the Tate curve.

A surprisingly deep and essential manifestation of tropical geometry in the study of integrable systems was shown in the talk of the prominent expert in the field R. Inoue, who worked out an example of box-ball system. Another nice example, a sandpile system, was presented by young researchers M. Shkolnikov and N. Kalinin, who showed that solutions to a certain sandpile model are perfectly described by tropical varieties. The talk by V. Fock brought together integrable systems, geometry of flag varieties and tropical in nature combinatorics, which resulted in very promising and interesting structures.

Problems of real algebraic and symplectic geometry linked to tropical geometry and amoebas appeared in talks by E. Brugallé, T. de Wolff and A. Renaudineau. So, E. Brugallé extended the notion of a (simple) Harnack curve to the case of pseudo-holomorphic curves and showed that their isotopy classification coincides with that for algebraic Harnack curves. T. de Wolff exhibited an efficient criterion of positivity of real polynomials via the geometry of their amoebas. A. Renaudineau applied the patchworking construction to the study of topology of real surfaces in toric three-folds.

We hope that the very intensive and substantial exchange of a broad spectrum of ideas during the workshop will stimulate the further research in the main discussed problems, which still are far from being completely settled.

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