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Mini-Workshop: Discrete p-Laplacians: Spectral Theory and Variational Methods in Mathematics and Computer Science

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Abstract. The p-Laplacian operators have a rich analytical theory and in the last few years they have also offered efficient tools to tackle several tasks in machine learning. During the workshop mathematicians and theoretical computer scientists working on models based on p-Laplacians on graphs and manifolds have presented the latest theoretical developments and have shared their knowledge.

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Introduction by the Organisers

The mini-workshop *Discrete p-Laplacians: Spectral Theory and Variational Methods in Mathematics and Computer Science* organized by Matthias Hein, Daniel Lenz, and Delio Mugnolo was involved with a field common to both mathematics and computer science in recent investigations around the so-called discrete p-Laplacian. The participants came from various parts of mathematics, including geometry and analysis, and computer science with a slight majority of participants from mathematics. A central aim was to make the involved researchers aware of methods and objectives of the 'other' community. Accordingly, the talks were mostly introductory in nature and presented a wide range of topics within the field. Special attention was paid to

- classical $p = 2$ theory for graphs (as the starting point for the case of $p \neq 2$),
- classical $p = 2$ and $p \neq 2$ theory for manifolds,

- Cheeger cuts in both mathematics and computer science,
- discretization procedures to get from continuum models to discrete models and vice versa.

By giving a brief introduction into the topic, we will now put these points in perspective. The *p-Laplacian* operators

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \qquad p > 1 \ ,$$

have been studied since the 1960s in order to model special diffusive systems. One decade later Yamasaki proposed a discrete version of them: because the signed incidence matrix $\mathcal{I}$ of a directed graph can be seen as a discrete version of the divergence operator, a *discrete p-Laplacian* operator is naturally defined by

$$\mathcal{L}_p f := \mathcal{I}(|\mathcal{I}^T f|^{p-2} \mathcal{I}^T f), \qquad p > 1 \ .$$

The special case $p = 1$ where the discrete p-Laplacian becomes multi-valued is of particular interest.

It was mostly the potential theoretical features of this family of difference operators that initially motivated their study. However, it was observed in the early 1990s by Perona, Malik, P.-L. Lions and other authors that the parabolic equation associated with Δ_p be conveniently used for image processing. Given a picture, i.e., a function $u_0 : \Omega \to \mathbb{R}^k$ ($k = 1$ or $k = 3$ for a b/w or rgb picture, respectively), the rationale behind the choice of parameter p relies upon the modeling purposes, as diffusion-driven smoothing of input pictures will be stronger in regions of *low* gradient for $1 \leq p < 2$, but in regions of *high* gradient for $2 < p < \infty$: this suggests applications to denoising or segmentation ($p \approx 1$) or morphing ($p \approx \infty$), respectively. Observe that

$$\frac{\partial f}{\partial t}(t, \mathsf{v}) = -\mathcal{L}_p f(t, \mathsf{v}), \qquad t \geq 0, \ \mathsf{v} \in \mathsf{V} \ , \tag{1}$$

turns into a partial differential inclusion for $p = 1$, as $\mathcal{L}_1$ is multivalued. Plugging a noisy picture as the initial data of (1) for $1 \leq p < 2$ and letting the system evolve with respect to the fictive time variable t will expectedly deliver pictures that are less and less blurry.

These considerations have paved the road for the celebrated Rudin–Osher–Fatemi model of image denoising, which is essentially an optimization problem for the energy functional

$$\mathcal{E}_p : u \mapsto \frac{1}{p} \int_\Omega |\nabla u|^p dx$$

associated with Δ_p, for $p \approx 1$. Analogous considerations hold for the discrete p-Laplacians.

The discrete p-Laplacians are used for similar reasons for clustering purposes: given a set of data, i.e., of vectors in $\mathbb{R}^d$, a graph is built upon determining an adjacency structure and hence a graph by means of a similarity function. Mirroring the structure of $\mathcal{L}_p$ for $p \to 1$, its eigenvectors will be strongly localized: the supports of its positive and negative parts will deliver meaningful clusters of the graph and thus is used to detect structures in a graph in an unsupervised way in machine learning.

If the graph is finite, then 0 is an eigenvalue of $\mathcal{L}_p$ for all $p \in [1, \infty)$. Strictly positive eigenvalues yield interesting information about the Cheeger constant, which is defined as

$$h_\rho(\mathsf{G}) := \min_{S \subset \mathsf{V}} \frac{|\partial S|}{\min\{|S|, |S^C|\}} ,$$

where ∂S denotes the set of edges with one endpoint in a subset S of the vertex set V and the other in its complement S^C, and $|\cdot|$ is the measure of a set with respect to a given node weight ρ. The famous isoperimetric inequality relating the second eigenvalue of the classical Laplacian ($p = 2$) has first been established by Cheeger for Riemannian manifolds and then for graphs by Alon and Milman. In computer science the so called spectral relaxation of the Cheeger cut problem – which is known to be NP-hard – has been used for clustering the vertices of a graph. Subsequently, similar inequalities have been established for the p-Laplacian both for the continuous and discrete problem. The case $p = 1$ is particularly interesting as the second eigenvalue of the discrete 1-Laplacian is equal to the Cheeger constant and the second eigenvector is the indicator vector of the optimal partition.

We quickly summarize the talks and their relation to the above subjects.

The relation of discrete and continuous graph Laplacian has been discussed in talks by Y. Kurylev, M. Gerlach and D. Slepcev. Y. Kurylev showed that eigenvalues and eigenvectors of the Laplace-Beltrami operator of a compact Riemannian manifold can be approximated by the corresponding objects of the discrete graph Laplacian built on a ϵ-net of the manifold. M. Gerlach showed that this approximation works as well for the case where the discretization is built from an i.i.d. sample of the manifold. D. Slepcev discussed Gamma-convergence of the Cheeger cut corresponding to the the second eigenvalue of the 1-Laplacian of a neighborhood graph built from an i.i.d. sample of a compact Riemannian manifold to the corresponding Cheeger cut of the manifold. J. Giesen gave a talk on spectral embeddings via particularly constructed graphs and their application in exploratory data analysis.

The relation of Cheeger cuts and discrete p-Laplacian and generalizations of the classical Cheeger inequality were discussed by M. Hein, S. Liu and D. Zhang. M. Hein discussed the relation of the spectrum of the p-Laplacian and (higher-order) Cheeger cuts and discussed generalizations to directed graphs and hyerpgraphs with applications in machine learning. S. Liu discussed generalization of the classical case $p = 2$ to signed graphs and more general magnetic Laplacians and gave Cheeger inequalities both for continuous and discrete case. D. Zhang discussed the graph 1-Laplacian and its properties in particular also of higher-order eigenvectors. M. Keller showed how powerful Cheeger inequalities for general p could be obtained for infinite graphs with unbounded degree via intrinsic metrics. J. Kerner extended these ideas to the case of p-Laplacians on quantum graphs, a possible relaxation of the usual combinatorial graph setting.

The topic of intrinsic metrics in the description of diffusion processes was taken up by D. Lenz and put in the context of general Dirichlet forms, which cover Laplacians on graphs and manifolds. The classical topic of the Feller property for diffusion was presented for manifolds by A. Setti and for graphs by R. Wojciechowski. S. Golénia showed how rich the theory of such a simple object as the diagonal matrix of vertex degrees can be and developed a comprehensive operator theory thereof. D. Mugnolo studied well-posedness, long-time behaviour and regularity of the solutions of the parabolic differential equation associated with discrete p-Laplacians on graphs and hypergrahs by means of a nonlinear extension of the theory of Dirichlet forms. B. Kawohl presented an overview on the theory of eigenvalues, eigenfunctions and nodal domains of p-Laplacians on domains, while P. Pucci offered an invitation to recent results on Kirchoff-type evolution equations associated with the fractional p-Laplacians.

The great atmosphere of Oberwolfach lead to numerous discussions and to a fruitful mutual exchange of ideas and concepts of the participants which had a quite heterogeneous background in mathematics and computer science. Thus we think that the mini-workshop has been very successful in partially initiating and partially strengthening the interaction between mathematics and computer science in all aspects around continuous and discrete aspects of the p-Laplacian. The organizers would like to thank the administration and staff members of the Oberwolfach institute for their hospitality and the great support before and during the workshop.

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