

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 2/2015

DOI: 10.4171/OWR/2015/2

New Discretization Methods for the Numerical Approximation of PDEs

Organised by
Stephan Dahlke, Marburg
Gitta Kutyniok, Berlin
Rob Stevenson, Amsterdam
Endre Süli, Oxford

11 January – 17 January 2015

ABSTRACT. The construction and mathematical analysis of numerical methods for PDEs is a fundamental area of modern applied mathematics. Among the various techniques that have been proposed in the past, some – in particular, finite element methods, – have been exceptionally successful in a range of applications. There are however a number of important challenges that remain, including the optimal adaptive finite element approximation of solutions to transport-dominated diffusion problems, the efficient numerical approximation of parametrized families of PDEs, and the efficient numerical approximation of high-dimensional partial differential equations (that arise from stochastic analysis and statistical physics, for example, in the form of a backward Kolmogorov equation, which, unlike its formal adjoint, the forward Kolmogorov equation, is not in divergence form, and therefore not directly amenable to finite element approximation, even when the spatial dimension is low). In recent years several original and conceptionally new ideas have emerged in order to tackle these open problems.

The goal of this workshop was to discuss and compare a number of novel approaches, to study their potential and applicability, and to formulate the strategic goals and directions of research in this field for the next five years.

Mathematics Subject Classification (2010): 35C20, 41A25, 41A65, 42C40, 65N12, 65T60, 65F20.

Introduction by the Organisers

The workshop *New Discretization Methods for the Numerical Approximation of PDEs* was organized by Stephan Dahlke (Marburg), Gitta Kutyniok (Berlin), Endre Süli (Oxford), and Rob Stevenson (Amsterdam). This meeting was attended by 51 participants from 11 countries.

Numerical approximation of PDEs is one of the central areas of computational mathematics, stimulated by the multitude of applications of PDEs in mathematical models in the sciences, engineering and economics. There is a rich arsenal of numerical techniques for PDEs, including finite difference methods, finite element methods, finite volume methods, spectral methods, and wavelet methods, to name just a few. The finite element method (FEM), in particular, has been successfully applied to linear and nonlinear PDEs in, typically, conservative form, that arise in continuum mechanics, such as the fundamental partial differential equations of solid and fluid mechanics. Some of the noteworthy features of finite element methods include their applicability to a wide class of problems, their convenience in terms of the use of locally refined computational grids and local variation of the polynomial degree in the finite element space (as in h -version and (h, p) -version adaptive FEMs), and their flexibility in representing computational domains possessing complicated geometries. By now, powerful software packages based on finite element methods have been developed and it is fair to say that the theory of FEMs is an established and mature field of numerical analysis. While it is expected that the range of applications of FEMs will continue to grow (a recent active area being, for example, the development and mathematical analysis of FEMs for geometric PDEs and PDEs on manifolds), it seems unclear whether ground-breaking new theoretical contributions are likely to emerge in the subject.

Contemplating scientific challenges that have arisen in recent years, it is possible to identify problem classes for which the performance of existing numerical techniques (and finite element methods in particular) is not entirely satisfactory. These include PDEs whose solutions develop singularities along lower-dimensional manifolds (e.g. blow-up phenomena in combustion problems and in kinetic models of chemotaxis (Keller–Segel system)), nonlinear hyperbolic conservation laws, for which smooth initial data can evolve into solutions that contain discontinuities (shocks and contact discontinuities), and transport-dominated diffusion equations whose solutions exhibit thin internal and boundary layers. One of the key open questions is in particular whether an adaptive finite element approximation of an elliptic or parabolic transport-dominated diffusion equation realizes the convergence rate that could be obtained with the best possible partition from the class of all partitions generated by, say, the newest vertex bisection technique; as a matter of fact, the convergence rate that could be obtained even with the best possible partition is unlikely to be “optimal” for such equations because of the loss of regularity of the analytical solution (in the scale of Besov spaces relevant for isotropic approximations) in the limit of the Péclet number tending to $+\infty$. Consequently, “anisotropic refinement” techniques will be needed in order to achieve the rate that is “optimal” for a given polynomial degree used in the finite element space.

Recently, several original and conceptionally novel ideas have been proposed whose potentials and scope of applicability are still being investigated. Among those are (adaptive) numerical schemes based on anisotropic ansatz functions, mixed dictionaries/frames or tensor wavelets. Other exemplary classes are low-rank tensor techniques for high-dimensional PDEs, schemes based on compressed sensing, meshless methods, and reduced-basis methods. The main focus of this workshop was to investigate the potentials of these newly developed discretization schemes and to identify and manifest promising future research directions in the field.

The workshop featured 36 talks, thereof 11 longer overview talks. Some highlights of the presentations include:

- **Reduced basis methods:** Wolfgang Dahmen reported on a double greedy algorithm for solving radiative transfer problems. To reduce the dimension of the problem, the angular variables are treated as parameters, and the solution manifold is approximated by the reduced basis method. The expansion coefficients in this basis are determined by solving stable Petrov-Galerkin problems in the reduced space. The test spaces are generated through an interior greedy loop.
- **Low rank tensors:** Reinhold Schneider gave an overview of low rank tensor techniques for the numerical solution of high dimensional PDEs. The recently introduced Hierarchical Tucker tensor formats and Tensor Trains offer stable and robust approximations at low cost. Approximations in those formats can be found by applying matrix low rank factorisation techniques (SVD).
- **Entropy-stable finite difference schemes:** Eitan Tadmor's lecture focussed on the importance of entropy stability in the dynamics of nonlinear systems of conservation laws and related convection-diffusion equations. He presented a general theory of entropy stability for difference approximations of such nonlinear partial differential equations, and illustrated the general theory through a range of first- and second-order accurate finite difference schemes for a variety of scalar problems as well as entropy stable schemes for the Euler and Navier–Stokes equations. Recent computations of entropy-measure-valued solutions based on a class of arbitrarily high order accurate and entropy stable TeCNO schemes were also shown.
- **Ridgelet and shearlet based discretisations for transport problems and wave propagation:** Philipp Grohs presented a ridgelet-based discretization of the kinetic transport equation. Using either a sparse collocation approach in the transport direction, or a tensor product construction, the system was solved in optimal complexity, even in the presence of line singularities in the solution.

The numerical solution of inverse scattering problems was discussed by Philipp Petersen. In the two-dimensional case, scatterers can be modeled by curves. These curves have optimally sparse representations in shearlet systems. This suggests to solving the inverse problem with a sparsity

promoting Tikhonov regularization term. The approach generalizes to certain linearized inverse problems.

The organizers would like to take the opportunity to thank MFO for providing support and a very inspiring environment for the workshop. The magic of the place (as coined by one of the participants) and the pleasant atmosphere contributed greatly to the success of the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: New Discretization Methods for the Numerical Approximation of PDEs

Table of Contents

Assyr Abdulle	
<i>Reduced basis techniques for multiscale methods</i>	95
Markus Bachmayr (joint with Albert Cohen)	
<i>Kolmogorov widths and low-rank approximations of parametric elliptic PDEs</i>	98
Sören Bartels	
<i>Numerical approximation of total-variation regularized problems</i>	101
Kolja Brix (joint with Martin Campos Pinto, Claudio Canuto, and Wolfgang Dahmen)	
<i>Multilevel Preconditioning of Discontinuous-Galerkin Spectral Element Methods</i>	102
Petru A. Cioica (joint with S. Dahlke, K.-H. Kim, S. Kinzel, K. Lee, F. Lindner, T. Raasch, K. Ritter, and R.L. Schilling)	
<i>Regularity of Stochastic Partial Differential Equations in Besov Spaces Related to Adaptive Schemes</i>	106
Wolfgang Dahmen (joint with Christian Plesken)	
<i>Greedy approximation and radiative transfer</i>	109
Laurent Demanet	
<i>Randomized sampling of matrices: three recent results</i>	111
Lars Diening (joint with Christian Kreuzer)	
<i>Instance optimality of the maximum strategy</i>	114
Massimo Fornasier (joint with Marco Artina, Stefano Micheletti, and Simona Perotto)	
<i>Anisotropic mesh adaptation for crack detection in brittle materials</i>	117
Silke Glas (joint with Karsten Urban)	
<i>Reduced Basis Approximation of Noncoercive Variational Inequalities</i> ..	119
Philipp Grohs (joint with Axel Obermeier)	
<i>Ridgelet Discretization of Linear Transport Equations</i>	121
David P. Hewett (joint with Simon Chandler-Wilde, Stephen Langdon, Ashley Twigger, Samuel Groth, Markus Melenk)	
<i>Hybrid numerical-asymptotic boundary element methods for high frequency wave scattering</i>	122

Helmut Harbrecht (joint with Christoph Schwab and Johannes Tausch) <i>Sparse BEM for the heat equation</i>	124
Paul Houston (joint with Paola Antonietti, Andrea Cangiani, Emmanuil Georgoulis, and Stefano Giani) <i>hp-Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes</i>	126
Daniel Kressner (joint with Luka Grubišić, André Uschmajew) <i>On low-rank approximability of solutions to operator equations and eigenvalue problems</i>	128
Angela Kunoth (joint with Christoph Schwab) <i>Adaptive Approximations of Parametric PDE-Constrained Control Problems</i>	130
Tony Lelièvre <i>Recent results on greedy algorithms and the reduced basis method for high-dimensional partial differential equations</i>	134
Dominik Lellek (joint with Stephan Dahlke, Shaun Lui, Rob Stevenson) <i>Adaptive Wavelet Schwarz Methods for Nonlinear PDEs</i>	137
Volker Mehrmann (joint with C. Carstensen, J. Gedicke, A. Miedlar, S. Quaraishi, C. Schröder) <i>A global view on adaptivity and sparsity in PDE discretization</i>	139
Mario Ohlberger (joint with Kathrin Smetana) <i>A Dimensional Reduction Approach Based on the Application of Reduced Basis Methods in the Framework of Hierarchical Model Reduction</i>	141
Ivan Oseledets <i>Solution of high-dimensional partial differential equations using tensor decompositions</i>	144
Peter Oswald (joint with Michael Griebel, Weiqi Zhou) <i>Schwarz Iterative Methods: New Developments</i>	146
Philipp Petersen (joint with Gitta Kutyniok, Volker Mehrmann) <i>Regularization and Numerical Solution of Inverse Scattering Problems using Shearlet Frames</i>	149
Bojan Popov (joint with Jean-Luc Guermond, Murtazo Nazarov, and Yong Yang) <i>Recent Developments in Entropy Viscosity</i>	152
Thorsten Raasch (joint with Stephan Dahlke, Peter Oswald) <i>Quarkonial frames and their connection to high-order schemes</i>	154
Reinhold Schneider <i>Hierarchical tensor representations and tensor networks</i>	157

Christoph Schwab (joint with J. Dick, F.Y. Kuo, T. LeGia, I.H. Sloan)	
<i>High Order, Quasi Monte-Carlo Petrov-Galerkin Discretization for</i>	
<i>countably-parametric operator equations</i>	160
Iain Smears (joint with Endre Süli)	
<i>Discontinuous Galerkin finite element approximation of</i>	
<i>Hamilton–Jacobi–Bellman equations with Cordes coefficients</i>	162
Benjamin Stamm (joint with Eric Cancès, Geneviève Dusson, Yvon	
Maday, Martin Vohralik)	
<i>A perturbation-method-based post-processing of planewave approximations</i>	
<i>of nonlinear Schrödinger equations</i>	165
Eitan Tadmor	
<i>From Perfect Derivatives to Conservative Differences</i>	166
Alexandra Tcheng (joint with Jean-Christophe Nave)	
<i>A fast-marching method for non-monotonically evolving fronts</i>	170
Manuela Utzinger (joint with Helmut Harbrecht)	
<i>Adaptive Wavelet BEM</i>	172
Thomas P. Wihler	
<i>hp-Discontinuous Galerkin Time Stepping Discretizations</i>	175
Harry Yserentant (joint with Stephan Scholz)	
<i>The approximation of wavefunctions by anisotropic Gauss functions</i> ...	178