Mathematisches Forschungsinstitut Oberwolfach

Report No. 27/2014

DOI: 10.4171/OWR/2014/27

Okounkov Bodies and Applications

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25 May – 31 May 2014

Abstract. The theory of Newton-Okounkov bodies, also called Okounkov bodies, is a relatively new connection between algebraic geometry and convex geometry. It generalizes the well-known and extremely rich correspondence between geometry of toric varieties and combinatorics of convex integral polytopes. Following a successful MFO Mini-workshop on this topic in August 2011, the MFO Half-Workshop 1422b, "Okounkov bodies and applications", held in May 2014, explored the development of this area in recent years, with particular attention to applications and relationships to other areas such as number theory and tropical geometry.

Mathematics Subject Classification (2010): 14C20, 14M25, 51M20, 14N25 , 14T05.

Introduction by the Organisers

Okounkov bodies were first introduced by Andrei Okounkov, in a construction motivated by a question of Khovanskii concerning convex bodies govering the multiplicities of representations. Recently, Kaveh-Khovanskii and Lazarsfeld-Mustata have generalized and systematically developed Okounkov's construction, showing the existence of convex bodies which capture much of the asymptotic information about the geometry of (X, D) where X is an algebraic variety and D is a big divisor. This theory of Newton-Okounkov bodies can be viewed as a vast generalization of the well-known theory of toric varieties. The study of Okounkov bodies is a new research area with many open questions, and the purpose of the Half-Workshop 1422b *Okounkov bodies and applications*, organised by Megumi Harada (McMaster), Kiumars Kaveh (Pittsburgh), and Askold Khovanskii (Toronto), was

to explore the many recent (and potential new) applications of this theory to other research areas.

The Half-Workshop was well attended with over 20 participants, with broad geographic representation from all continents. The group of participants was a nice blend of researchers with various backgrounds such as tropical geometry, representation theory, toric topology, symplectic topology, integrable systems, and number theory. In addition to the senior participants, there were 2 participants supported through the Oberwolfach Leibniz Graduate Students" program. The workshop consisted of 18 research talks in total.

In the remaining part of this introduction we briefly describe some of the topics discussed at the workshop.

One of the major themes of the workshop was to define functions to and from Newton-Okounkov bodies. Functions *from* Newton-Okounkov bodies were discussed by Alex Kuronya, with a view towards applications in the study of big divisors and positivity of linear series on algebraic varieties. Functions *to* Newton-Okounkov bodies were discussed by David Witt-Nystrom in his talk on joint work with Julius Ross, in which they define a kind of analogue of a 'moment map' to a Newton-Okounkov body. (Witt-Nystrom also gave another talk on transforming metrics of a line bundle which provided some background on his work on moment maps.)

Several of the talks reported on recent progress in the theory of Newton-Okounkov bodies. Victor Lozovanu reported on recent joint work with Kuronya on positivity of linear series on surfaces, and Kiumars Kaveh presented joint work with Khovanskii on the theory of *local* Newton-Okounkov bodies. One of the junior participants Takuya Murata, invited through the Oberwolfach Leibniz Graduate Students program, was given the opportunity to present his Ph.D. thesis results (supervised by one of the organizers, Kiumars Kaveh) on the asymptotic behavior of multiplicities of reductive group actions.

Another major purpose of the workshop was to explore possible connections with other research areas. In this spirit, Huayi Chen gave a talk outlining possible avenues of applications of Newton-Okounkov bodies to arithmetic. Similarly, Sam Payne gave a talk on tropical methods for the study of linear series and Buchstaber gave a presentation on $(2n, k)$-manifolds; in both talks, many themes overlapped with those arising in the study of Newton-Okounkov bodies. Furthermore, Boris Kazarnovskii talked about an extension of the theory of Newton-Okounkov bodies to the non-algebraic setting of *exponential* sums, and about a very surprising relation of this non-algebraic subject to modern algebraic geometry.

Symplectic geometry, symplectic topology, integrable systems, and toric degenerations also played a main role in the workshop. In this direction, both Chris Manon and Johan Martens reported on their recent work on the Vinberg monoid, while Yuichi Nohara gave a talk on toric degenerations of integrable systems on the Grassmannian and an application to the computation of the potential functions arising in symplectic topology. Continuing the theme of symplectic topology, Kaoru Ono gave a talk on Lagrangian tori in $S^2 \times S^2$.

The relation between Newton-Okounkov bodies and Schubert calculus was also a strong theme of the workshop. Valentina Kiritchenko spoke about a 'geometric mitosis' operation on Newton-Okounkov polytopes (associated to flag varieties) which give rise to collections of faces of the polytope representing a Schubert cycle. Vladlen Timorin gave a talk on counting vertices of Gel'fand-Cetlin polytopes, which are a special case of Newton-Okounkov bodies of flag varieties. June Huh presented his results on positivity of Chern classes of Schubert cells and varieties, concluding with open questions in this area which relate to Newton-Okounkov bodies. Finally, in the last talk of the workshop, Dave Anderson spoke on computing the effective cone of Bott-Samelson varieties, which arise naturally in the study of Newton-Okounkov bodies due to their central role in representation theory and the geometry of flag varieties.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

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