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Mini-Workshop: Mathematical Physics meets Sparse Recovery

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Abstract. In recent years, there have been several fruitful interchanges of methods between the fields of sparse and low-rank recovery on the one hand and quantum information theory on the other hand. One way to understand this seemingly surprising coincidence is that the analysis of vector- and matrix-valued randomized constructions plays an important role in both fields. An example is the realization that certain matrix-valued large deviation bounds can be employed to substantially simplify and generalize the analysis of low-rank matrix recovery schemes.

In this workshop, the participants worked to identify and collaborate on further mathematical problems that are being researched in parallel by the two communities. Topics that have been discussed include

- Tools for the analysis of vector- and matrix-valued randomized constructions and their application to phase retrieval problems
- Conversely, tools for *de*-randomizing such protocols, based, e.g., on spherical designs.
- Uncertainty relations, e.g., for the task of lower-bounding the number of measurements required for signal identification.
- Time-frequency methods (known as phase-space methods in physics).
- Matrix- and tensor norms: computational tools, complexity, relaxations and their application to tensor recovery.

Mathematics Subject Classification (2010): 15xx, 15A83, 15A42, 47A52, 46N50.

Introduction by the Organisers

The mini-workshop *Mathematical Physics meets Sparse Recovery*, organised by David Gross (Freiburg), Felix Krahmer (Göttingen), Rachel Ward (Austin), and Andreas Winter (Bellaterra) successfully initiated intensive communication between researchers in two different areas. Among the participants there were, on the one hand, nine researchers focusing on mathematical signal processing, in particular on sparse recovery and related questions, and, on the other hand, seven mathematical physicists who have contributed to quantum information theory. The connection between these fields first started to arise with the works of David Gross, one of the workshop organizers, who demonstrated that certain problems in quantum information theory are closely related to problems in low-rank matrix recovery [1]. So tools developed in mathematical physics can help solve mathematical signal processing problems. Following up on his results, a collaboration between David Gross, his PhD student Richard Küng, also a workshop participant, and Felix Krahmer, another organizer, started [4], strengthening the connection between the fields. This collaboration laid the foundation for the mini-workshop.

From their collaboration experience, they realized that one of the biggest obstacles to an intensified collaboration between the fields was the often quite different mathematical language and notation used in the two fields, it was decided that a substantial part of the workshop should be devoted to communicating to the respective other group the way of thinking and the terminology employed. Another substantial part was to be devoted to communicating the general research goals to find collaboration projects, and the third intended pillar was organized discussions in small groups.

The workshop started by two introductory talks of about 90 minutes each by participants representing the sparse recovery community. Dustin Mixon gave a general overview of mathematical signal processing problems on a very broad level and Holger Rauhut continued by providing a more technical introduction to the methods currently used. The goal of these two talks was to communicate the type of problems arising in the signal processing community to those participants from quantum information theory who were experts on relevant methods while not having applied them to mathematical signal processing before. Reciprocating, Andreas Winter gave a similar 90 minute talk introducing the problems and mathematical methods at the center of quantum information theory.

After these introductions, the workshop proceeded as follows: days were kicked off by a one hour talk requested the previous day. After that, the participants were divided into collections of small, mixed groups that started working on specific questions. Topics that received significant attention by those groups are as follows:

Tight large-deviation bounds for sums of random matrices. This theory was essential to set up the connection between the two communities. Originally developed by A. Winter to treat quantum information problems, large-deviation bounds for sums of random matrices were introduced into the theory of sparse and low-rank recovery by D. Gross and later greatly refined at the hands of J. Tropp. Joel Tropp presented the state of the art and the most pressing problems as he sees the theory. A concrete problem – removing a sometimes spurious logarithmic dimension factor in front of the generally exponentially small large deviation probability – created particular interest. While the problem was not solved during the workshop, collaborations have been initiated as a result.

A conjectured "matrix arithmetic-geometric mean inequality." Recently, a noncommutative extension of the well-known arithmetic-geometric mean inequality for vectors was proposed by B. Recht and C. Ré [2], and this conjecture was communicated to the workshop participants by Felix Krahmer and Rachel Ward. Such an inequality is of interest in machine learning and signal processing as it would give theoretical justification to the observed effect that *without-replacement* sampling schemes outperform *with-replacement* sampling schemes in randomized sequential optimization algorithms. Although this inequality has been verified in certain special cases, neither a general proof nor a counter-example has been found. While the conjecture was not resolved during the workshop, Marius Junge made initial discoveries towards finding a counter-example to the conjecture using free probability and recent techniques he had devised for related matrix problems.

The nascent theory of tensor recovery. After having treated sparse vectors and low-rank matrices, the signal analysis community has recently turned their attention to the theory of learning low-rank tensors from underdetermined measurements. Tensor problems likewise appear in quantum mechanics, where many-body wave functions are just elements in large tensor spaces. After extremely well-received talks by Y.-K. Liu about applications in natural language processing and by Z. Stojanac on recent ideas for tensor norm relaxations based on theta bodies, several discussions ensued. A mixed group tried to devise measurements that are incoherent w.r.t. all low-rank tensors – generalizing previous such constructions that the quantum community routinely uses for matrices. The prospect of proving results for completely symmetric tensors was discussed. V. Cevher introduced the physicists to dual smoothing techniques that might speed up numerical methods for computing tensor norms. There was an extensive discussion as to whether tests for quantum separability based on symmetric extensions could give rise to well-performing convex proxies for tensor rank.

Phase space and time-frequency methods. Inspired by an earlier talk by G. Pfander on time-frquency analysis and sparse recovery, D. Gross presented unpublished work on phase space support-rank uncertainty relations and their role in proving lower bounds to low-rank matrix recovery problems. A collaboration between Gross and Pfander was initiated with the goal to generalize these results from discrete to continuous Gabor systems.

Stability of PhaseLift. The ill-defined inverse problem of retrieving an unknown complex vector from "amplitude" measurements – i.e. linear measurements that are ignorant towards complex phases – has received considerable attention in the field of mathematical signal processing over the last few years. Moreover, problems of this type are also of considerable interest for doing quantum state tomography – an important subfield of quantum information theory [5]. During the course of the workshop, H. Rauhut and R. Küng started to tackle the important problem of proving stability guarantees for PhaseLift in the presence of noise for certain measurement setups – most notably random coded diffraction patterns [6]. Although the problem could not be fully solved during the workshop, partial results have been obtained and yet another ongoing collaboration (between H. Rauhut, R. Küng and D. Gross) arose. In this context, also a connection in the other direction arose, namely F. Krahmer and Y.-K. Liu started a collaboration with the goal to apply mathematical tools developed and applied in the context of phase retrieval problems to problems in quantum information.

Statistical trade-offs in low-rank recovery. It had been observed numerically [5] that the performance of low-rank based estimators for quantum state tomography displays the following behavior: Assume the total number of experimental samples taken is constant. There is the freedom to use these samples to either estimate a few linear functions of the unknown low-rank matrix to a high precision, or many distinct functions more coarsely. Maybe surprisingly, it turns out that the performance of the estimator is largely independent of that choice. This is of relevance to physical experiments – but the physicists failed to find a theoretical explanation for this behavior. M. Gutta suggested an approach based on asymptotic properties of the maximum likelihood estimator, while R. Saab proposed to use results about the behavior of right-inverses of near-isometric embeddings to attack the problem. Both ideas seem promising, and a collaboration between the aforementioned researchers, D. Gross and A. Winter to further pursue these questions has been initiated.

To conclude, this mini-workshop was by all accounts a great success. Through this meeting, certain mathematical language and notation barriers were overcome, and each community became aware of new problems and common interests among the other community, as well as new applications to problems and techniques already known. At the same time, several new collaborations formed between researchers in mathematical signal processing and quantum information theory, and concrete results to open problems have already been established. We expect a number of papers to come out of this meeting in the coming years.

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