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Mini-Workshop: Constructive Homological Algebra with Applications to Coherent Sheaves and Control Theory

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Abstract. The main objective of this mini-workshop is to bring together recent developments in constructive homological algebra. There, the current state already reached a level of generality which allows simultaneous application to diverse fields of applied and theoretical mathematics. In this workshop, we want to focus on simultaneous applications to system theory on the one side and to coherent sheaves and their cohomology on the other side. Surprisingly, these apparently remote fields share a considerable amount of common constructive methods. Bringing category theory and homological algebra to the computer leads to questions in logic and type theory. One goal of this workshop is to promote and enlarge this overlap.

Mathematics Subject Classification (2000): 14Qxx, 18Gxx, 13Dxx, 18E35, 18A40.

Introduction by the Organisers

The quest for constructivity in mathematics is as old as mathematics itself. Thanks to the emergence of powerful computers, constructivity is now regaining increased interest in several fields of applied and theoretical mathematics. System theory and algebraic geometry are apparently two remote representatives of such fields. However, as homological algebra is invading system theory, both fields now share a common powerful tool which turns out to be one of the keys to constructivity questions in both theories. A unification of the existing computational approaches is becoming necessary to extend the scope of applicability of constructive homological algebra developed by researchers in the different fields. The study

of such a unification also suggests a comparison between the different type systems underlying the computational models of existing computer implementations.

Modeling Abelian categories of coherent sheaves on non-affine schemes is more involved than modeling those of modules. The talks addressed recent results allowing a constructive description of Abelian categories of coherent sheaves on spaces with a finitely generated Cox ring S as a Serre quotient category of the category of finitely presented graded S-modules. This includes the constructive treatment of (local and) global Ext's of which sheaf cohomology is a particular case.

Introducing modules as an intrinsic description of linear functional systems in system theory opened the door to apply homological techniques stemming from algebraic analysis. This provides a unified framework for system theory, which expresses itself in terms of common concepts, techniques, results, algorithms, and even implementations. There are numerous module-theoretic properties with a system-theoretic interpretation, e.g., the module being torsion, torsion-free, pure, reflexive, projective, stably free, free. All these properties have system-theoretic counterparts, e.g., the existence of autonomous elements, of minimal (resp. injective or Monge) parametrizations, of Bézout identities. This makes their study crucial for applications in control theory and mathematical physics, e.g., motion planning, quadratic optimal control, solving variational problems, and searching for potentials or conservation laws, to name a few. Luckily, these module-theoretic properties can be described in terms of homological algebra using resolutions, extension and torsion functors, projective dimension, and the purity filtration.

The talks focused on new developments in constructive homological algebra with applications to

(1) coherent sheaves and their cohomology; equivariant vector bundles;
(2) system and control theory based on homological algebra techniques;
(3) discrete vector fields in algebraic topology.

One of the discussion sessions was devoted to the "univalent foundation of mathematics". Thierry Coquand, who attended the special year at the IAS, gave an informal talk about Voevodsky's new interpretation of dependent types where a type is thought of as a homotopy type. This interpretation suggests a new way to represent constructive mathematics in type theory. One remarkable feature of this formalism is that two algebraic structures that are isomorphic are equal which allows to transport properties from one structure to an isomorphic one.

Many participants profited from further stimulating discussion with participants of the two parallel mini-workshops "Spherical Varieties and Automorphic Representations" and "Localising and Tilting in Abelian and Triangulated Categories". For example, we would like to mention discussions with Stefan Schwede and Bernhard Keller about Morita theory, tilting theory, and (algebraically) triangulated categories.

This interdisciplinary mini-workshop was attended by 17 participants from different areas of mathematics: algebraic geometry, algebraic topology, constructive algebra, differential algebra, system and control theory, logic and type theory. The participants had enough time for several informal discussions about various

topics including Boij-Söderberg theory, equivariant vector bundles, symmetries of algebraic surfaces, discrete vector fields, and comparisons between the programing paradigms and implementations of homological algorithms in GAP4 and Coq. The organizers and the participants would like to thank the MFO for the excellent organization and the great hospitality. Christine Berkesch was funded by the "US Junior Oberwolfach Fellows" program of the NSF.

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