

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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## Graph Theory

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ABSTRACT. This was a workshop on graph theory, with a comprehensive approach. Highlights included the emerging theories of sparse graph limits and of infinite matroids, new techniques for colouring graphs on surfaces, and extensions of graph minor theory to directed graphs and to immersions.

*Mathematics Subject Classification (2010):* 05Cxx.

### Introduction by the Organisers

The aim of this workshop was to offer an exchange forum for graph theory and related fields in pure mathematics. We had eight ‘main’ longer talks, 22 shorter talks, seven informal workshops on topics suggested by the participants, and plenty of further informal interaction.

Particular emphasis was given to fields that have seen particularly exciting recent developments:

- Graph limits, either dense or sparse;
- Infinite matroids;
- Colouring graphs on surfaces;
- Graph minors, graph immersions, and tree-structure.

The theory of *graph limits*, initiated a few years ago by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi, has been taken up and developed further by a number of leading researchers. The original idea was to describe properties of dense graphs by continuous objects, *graphons*, that occur as limit objects of sequences of such graphs, so that analytical methods could be brought to bear on the study of

such properties. In an independent development, Razborov proposed a theory of *flag algebras*, whose applications to extremal graph homomorphism and induced subgraph density problems have turned out to be interchangeable with those of graph limits.

For (very sparse) graphs of bounded degree, such as Cayley graphs of finitely generated groups, Benjamini and Schramm had earlier developed another limit theory, also with probabilistic ingredients, but whose limits were essentially still graphs (albeit infinite). Extending these ideas, Elek recently proposed a more general notion of *graphings* as limit objects of sparse graphs.

Several of our talks were from this area, including the main talks by László Lovász on *Borel graphs, graphings, and limits of bounded-degree graphs*, and by Christian Reiher with a proof of *The clique density theorem*. This had been conjectured by Lovász and Simonovits in the 1970s.

Following the recent axiomatization of infinite matroids with duality by Bruhn, Diestel, Kriesell, Pendavingh and Wollan, infinite matroid theory has seen a surge of activity that has produced some deep results and conjectures, unifying some of its major open problems, and relating it to both infinite graph theory and logic.

Nathan Bowler gave a main talk about these developments. He put an emphasis on newly emerging bonds between infinite matroids and graphs topologized with their ends; these can be described elegantly in terms of the determinacy of infinite games. A highlight of the talk was a new packing/covering conjecture for infinite matroids, a central conjecture that unifies some of the main classical open conjectures about infinite matroids, such as matroid intersection and union.

The area of colouring graphs on surfaces is a classical one dating back to Heawood's formula from 1890. One chapter of its development was completed by the map color theorem of Ringel and Youngs in the 1960s. A modern approach to the subject and a research programme was initiated by Thomassen in the 1990s.

We had a main talk by Luke Postle on *Linear isoperimetric bounds for graph colouring*, a new technique in the area based on the recent discovery that many colouring results are a direct consequence of the fact that the corresponding 'critical' graphs (those that are minimally uncolourable) satisfy a certain isoperimetric inequality.

The theory of graph minors, initiated by Robertson and Seymour in the 1980s, is both maturing and expanding. It is maturing in that its main structural results are increasingly well understood: their assertions and proofs have been simplified and strengthened in a long process of identifying the essentials, converging to a leaner and more powerful theory that is now taking shape. It is also expanding: in the direction of directed graphs (in particular, of tournaments), of matroids, and of graph orderings stronger than that of minors, such as topological minors or immersions. There is now a unifying theory establishing the existence of canonical tree-decompositions of graphs and matroids that can distinguish their dense parts, such as higher-order blocks or tangles. Finally, graph minor theory is increasingly used in computer science, both for concrete algorithms and abstract complexity theory issues such as fixed-parameter tractability.

We had several talks on these topics, including main talks by Dániel Marx on *The  $k$ -disjoint paths problem in directed planar graphs*, and by Paul Seymour on *Grid immersion and multicommodity flows*.

Further main talks were given by János Pach on *Geometric graph theory* and Lex Schrijver on *The edge colouring model*.

Finally, we had a number of informal workshops focused on a particular topic, often initiated by the participants but open to all. Their topics were:

- Graph limits and flag algebras (convenor: Král)
- Ramsey goodness (convenor: Conlon)
- Infinite matroids (convenors: Bowler and Carmesin)
- Sparse regularity (convenors: Conlon and Fox)
- Flows (convenor: Šámal)
- Colin de Verdière - type parameters (convenor: van der Holst)
- The Lovász theta function (convenor: Šámal)

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