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Mini-Workshop: Topology of Real Singularities and Motivic Aspects

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30 September – 6 October 2012

Abstract. This workgroup focusses on some recent issues in real singularities, concerning the topology of the Milnor fibre of a singular map and several motivic aspects of singularities of sets definable in some structures over the reals or even over some valued field, with the ambition to develop the interplay between the two domains.

Mathematics Subject Classification (2000): 14B05, 14P25, 14P10, 32S55, 14E18.

Introduction by the Organisers

The workshop *Topology of Real Singularities and Motivic Aspects*, organised by Georges Comte (Université de Savoie) and Mihai Tibar (Université Lille I) was held 30 September – 6 October 2012. This workshop was well attended with 17 participants with broad geographic representation. 17 talks were given. The general topic of these talks was the topology of singular fibres of real mappings (or sets), in some tame category of sets such as algebraic sets, semi-algebraic sets, or sets definable in some structure over a given first order language.

• The existence of a stable fibration in the neighbourhood of an (isolated) singularity is fundamental for the understanding of the local structure of the space-function pair. In his well-known Princeton lecture notes, John Milnor studied the fibration of a complex analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. He explained the first steps of extending the study to a real analytic map germ $g : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with isolated singularity.

Many authors pursued ever since to implement and highly enrich Milnor's technique in other fields, such as non-isolated singularities of holomorphic functions, complete intersections with isolated singularities, singularities at infinity of polynomial mappings and meromorphic functions. Remarkably, the topological study of singularities and of their deformations gave new perspectives over a lot of other fields such as exotic structures, Lie algebras, mixed Hodge theory, equisingularity, Frobenius manifolds etc.

In the real setting, a recent progress took place by M. Oka's series of papers on "mixed singularities" i.e. real analytic map germs $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^2, 0)$. Oka develops Milnor's theory and finds real counterparts of some of his holomorphic results. For instance the existence of an open book fibration with binding the link $K = f^{-1}(0)$ is not insured like in the holomorphic setting, thus one researches the "best" conditions under which this happens, in the local setting and also in the global affine one. "Open book" appear under several names in different fields, for instance "Lefschetz fibrations" in algebraic and symplectic geometry, "fibered links" and "spinorial structures" in topology.

In the study of the numerous questions and conjectures which appeared, some interesting classes of singularities prove helpful: those with non-degenerate Newton principal part, and the radial and polar quasi-homogeneous singularities. We hope for instance that the Newton boundary techniques, locally and at infinity, would produce exotic examples like the clasical example by A'Campo in connexion to the Lefschetz number of the monodromy.

Along this path we aim to the study of the monodromy in the real setting, notably via the resolution by blow-ups and by toric methods.

• The second main stream of the workshop concerns motivic aspects of singularities.

To emphasize the perfect thematic continuity between the two domains into consideration during our workshop, one may start from A'Campo's formula for the Lefschetz number of the iterates of the monodromy of singularity in terms of the combinatorial data of a resolution of this singularity (including the Euler-Poincaré characteristic of the irreducible components of the exceptional divisor of the resolution). In the complex setting, this may be viewed as part of the theory of equisingularity, initiated by Zariski. In this context Zariski himself, then A'Campo, Briançon, Henry, Hironaka, Merle, Sabbah, Speder, Teissier, Lê-Dũng-Trang among others, established a number of results, relating different aspects of the geometry of the singularity (algebraic, topological, differential).

J. Denef and F. Loeser gave more recently a new point of view on this formula: in the spirit of Igusa's work in the p-adic framework, they proved that some formal series, the so-called motivic zeta function, built on some complex algebraic singularity, with coefficients in some formal Grothendieck ring of varieties, is a rational function and that the realization of this rational function via the Euler-Poincaré characteristic gives the Milnor number of the fibration. This rational function is therefore called the motivic Milnor fibre of the fibration. It would be crucial to completely understand the part of the topology in the proof of the rationality of the motivic Milnor fibre. Namely what kind of topological data is structurally encoded in the motivic Milnor fibre ? This issue contains for instance the so-called monodromy conjecture that aims to relate the poles of the motivic Milnor fibre and the eigenvalues of the monodromy function.

In this spirit, we would like to understand what topological invariants remain in the real motivic framework. Assuming that we would be able to establish a real substitute of the complex motivic Milnor fibre and prove a real version of the Denef-Loeser formula, is it still true that the Euler-Poincaré characteristic of semi-algebraic nearby fibres is contained in the topological and combinatorial data of a resolution? Indeed, in the real case, the combinatory of a resolution appears in a less trivial way after the possible realization of the real substitute of the complex motivic Milnor fibre and, consequently, a real formula probably would give more explicit views than in the complex case on the interplay between the topology of the nearby fibres and the combinatorial data of a resolution. To establish the real counterpart of the complex motivic Milnor fibre one has to find in particular the pertinent notion of Grothendieck ring of semialgebraic formulas.

The question underlying the question of a real motivic version of the complex Milnor fibre is the general problem of finding additive invariants on real algebraic or semi-algebraic sets, such as the Euler-Poincaré characteristic, the virtual Betti numbers etc. If this question gave rise to many successful attempts recently, much remains to do in this area.

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Table of Contents