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Mathematical Aspects of General Relativity

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Abstract. Mathematical general relativity, the subject of this workshop, is a remarkable confluence of different areas of mathematics. Einstein's equation, the focus of mathematical relativity, is one of the most fruitful nonlinear hyperbolic PDE systems under study. As well, some of the most challenging geometric analysis problems in Reimannian geometry and elliptic PDE theory arise from the study of the initial data for Einstein's equations. In addition, these studies play a crucial role in modeling the physics of astrophysical and cosmological systems. This workshop reflected the rapid progress seen in the field in recent years, and highlighted some of the most interesting questions under study in mathematical relativity.

Mathematics Subject Classification (2000): 83Cxx.

Introduction by the Organisers

Black holes and their stability: The black hole stability problem, a topic which has generated intense work in the past ten years by researchers spanning the areas of harmonic analysis, spectral theory, and partial differential equations all the way to theoretical physics, remained one of the main areas of activity in this year's workshop. The proof of quantitative boundedness and decay estimates for solutions of the wave equation on Kerr backgrounds in the entire sub-extremal range $|a| < M$, reported on in the previous 2009 Mathematical Aspects of Relativity workshop, was complemented in this year's workshop by the elucidation of the remaining extremal case, in which $|a| = M$. Stability results for this case were discussed, but even more surprisingly, *instability* results were also presented, in the talk of Stefanos Aretakis. Another new direction for black hole stability has been that of black holes with negative cosmological constant, an area marked by intense interest on the part of the string theory community. New results concerning Schwarzschild-AdS and Kerr-AdS spacetimes were reported in the talk of Jacques Smulevici, which included in particular a proof of logarithmic decay for solutions of the wave and Klein-Gordon equation, and, moreover, a proof that such rates are in fact sharp. The latter might suggest that *all asymptotically AdS space times are in fact non-linearly unstable*, a statement with important implications for high energy physics. This would extend the conjecture, made already at a 2007 Oberwolfach workshop on Analysis and Geometric Singularities, that pure-AdS space-time is unstable; this conjecture is now substantially supported by the recent numerical work which was presented in the talk of Piotr Bizon. (This numerical work features the very interesting phenomenon that the instability appears to be driven by a mechanism similar to fluid turbulence, in which energy is transported from low to high frequencies.) Yet another direction for the black hole stability problem has been to consider what happens in higher dimensions, where the zoology of stationary black holes is much richer. Harvey Reall reported on a very pretty application of the Penrose inequality to infer *non-linear instability* for certain higher dimensional black holes. This latter work was the original motivation for the result presented in Bob Wald's talk, which related a *linear stability*-type statement directly to the validity of a local Penrose inequality.

The above multifaceted progress on linear aspects of the stability problem makes a proof of the non-linear stability of the Kerr family (as solutions of the vacuum Einstein equations) an exciting prospect for the years to come. A new and different step in that direction was presented in the talk of Gustav Holzegel, which discussed a proof of the existence of a large class of vacuum spacetimes without symmetries which dynamically approach Schwarzschild or Kerr. These are constructed by imposing "scattering type data" on the horizon and on null infinity, and solving the Einstein equations "backwards".

Several other important issues involving black holes and other asymptotically flat space times were discussed at the meeting. Lydia Bieri presented extensions of Christodoulou's celebrated non-linear memory effect to the Einstein-Maxwell case, and showed in particular that non-linear memory has also a non-trivial contribution arising from the total electromagnetic radiation. Håkan Andréasson reported on a construction of spherically symmetric black hole solutions with collisionless matter arising from a complete regular past. Carsten Gundlach presented numerical evidence for a new type of critical behaviour at the threshold of immediate merger in binary black hole systems.

Cosmology:. Turning to mathematical problems arising in cosmology, Qian Wang spoke about a new local existence result in the spatially harmonic constant mean curvature (CMCSH) gauge. In particular, she described how to obtain local existence in H^s for $s > 2$. Needless to say, improved local existence results lead to improved breakdown criteria, and are therefore of interest in the context of proving global results.

Jared Speck presented joint work with Igor Rodnianski concerning big bang singularities in the scalar field and stiff fluid case. One particular consequence of the result is that perturbing initial data for a spatially flat FLRW solution leads to solutions that are similar to FLRW towards the past singularity. This constitutes the first known result which gives a detailed description of the singularity (including curvature blow up) for an open set of initial data. Juan Antonio Valiente Kroon presented joint work with C. Lübbe concerning future global non-linear stability of FLRW spacetimes containing a de Sitter-like cosmological constant and a radiation fluid.

Several important questions concerning oscillatory singularities in the spatially homogeneous setting remain unanswered. These concern the generic behaviour as well as the causal structure. In his talk, Alan Rendall presented joint work with S. Liebscher and S. Tchapnda on this topic. In particular, a construction of an unstable manifold of solutions converging to a heteroclinic cycle was described in the class of magnetic Bianchi VI_0 solutions.

Regularity: Issues related to causality theory for metrics of low regularity appear in many different contexts. In his talk, Piotr Chruściel discussed this topic, and presented a proof of the existence of a maximal globally hyperbolic development for sets of initial data with low regularity. Philippe LeFloch presented a general theory (developed together with several authors) for treating low regularity spacetimes with symmetries.

Another promising direction for future research is that which has been opened up by the study of interacting, impulsive gravitational wave-solutions of the vacuum equations, without symmetries, as discussed in the talk of Jonathan Luk. These singular solutions lie below the well-posedness threshold of curvature in L^2, and their existence is yet another manifestation of the remarkable structure which is present in the Einstein equations. The analysis of these solutions gives hope that spacetimes which have been conjectured to have even more severe singularities propagating on null cones, like those of generic vacuum black hole interiors, will soon be understood in complete generality.

Initial Data sets and their properties: The study of initial data sets which satisfy the Einstein constraint equations has been a major feature of all five of the Oberwolfach meetings on mathematical relativity. Many important issues involving this topic remain unresolved, and are currently very active areas of research. One of the long-standing issues is determining the extent to which the conformal method can be used to parametrize and construct the range of solutions of the Einstein constraint equations. For the constant mean curvature and near constant mean curvature solutions, this is well understood, but for others it is not. Romain Gicquaud discussed his recent results (with M. Dahl and E. Humbert) which provide a means for showing that, for certain sets of conformal data with non constant mean curvature, the conformal method does produce solutions. Another issue of long-standing interest is the positivity of the ADM mass in asymptotically flat initial data sets. While such positivity for $3+1$ dimensional spacetimes has been known for a number of years, it is known for higher dimensional spacetimes only for spin manifolds. Lan-Hsuan Huang, reporting on research done with M. Eichmair, D. Lee, and R. Schoen, showed that positivity holds for spacetime dimensions of $7 + 1$ dimension or less.

A key tool used in the work reported by Huang is the "marginally outer trapped surface", or "MOTS". These geometric structures play an increasingly important role in the study of initial data sets and their development. In Greg Galloway's talk, MOTS play a major role in the formulation and proof (with M. Eichmair and D. Pollack) of a new topological censorship theorem which uses criteria involving initial data only to restrict the allowed asymptotic topology of such data sets. The talk of Marc Mars (with M. Reiris) also discussed MOTS, focussing on the relationship between MOTS (locally defined) and black holes (globally defined) in stationary and static spacetimes.

Helmut Friedrich's talk focussed on issues related to asymptotic properties of asymptotically flat initial data sets. Working with time symmetric vacuum data, he studies the relationship between those data sets which are asymptotically conformal to static data and those which are asymptotically static. The talk of Martín Reiris dealt with an interesting phenomenon related to sequences of axisymmetric initial data sets for which an inequality relating areas and angular momenta is saturated. He shows (with S. Dain) that for such sequences, the limiting data set exhibits the phenomenon of an "extreme Kerr throat". The talk of Christine Sormani addressed the mathematics of convergence of Riemannian manifolds. She introduced the notion of "intrinsic flat convergence", and discussed ways in which this notion could be useful in studying initial data sets.

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