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Calculus of Variations

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Abstract. Since its invention, the calculus of variations has been a central field of mathematics and physics, providing tools and techniques to study problems in geometry, physics and partial differential equations. On the one hand, steady progress is made on long-standing questions concerning minimal surfaces, curvature flows and related objects. On the other hand, new questions emerge, driven by applications to diverse areas of mathematics and science. The July 2012 Oberwolfach workshop on the Calculus of Variations witnessed the solutions of famous conjectures and the emerging of exciting new lines of research.

Mathematics Subject Classification (2000): 49-xx, 35Jxx, 53Cxx, 58Exx.

Introduction by the Organisers

The workshop has been attended by 50 participants, 14 of which have completed their PhD studies recently or were still completing them at the time of the workshop. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and by emergent applications. This edition of the conference saw the recent solution of important famous conjectures and recorded big progresses in areas which have been stuck for a long time.

The main themes of the workshop can be roughly divided into five groups:

- The theory of area-minimizing and minimal surfaces;
- Optimal transportation and applications;
- Geometric variational problems;

- Variational problems from mathematical physics;
- Geometric flows.

The first group contains one highlight of the workshop: Simon Brendle's talk on his recent proof of the Lawson Conjecture, which characterizes the Clifford torus as the unique minimal embedded torus in the standard $\mathbb{S}^3$. The proof hinges on a virtuoso application of a technique, combining variable-doubling and a maximum principle argument, introduced by G. Huisken in the quite different setting of distance-comparison theorems for the curve-shortening flow, and further developed by B. Andrews in recent work on non-collapsing in mean curvature flow. The theory of minimal surfaces in Riemannian 3-manifolds was also the topic of Theodora Bourni's joint work with Baris Coskunuzer, where the authors study the genus of area-minimizing surfaces in compact orientable 3-manifolds M with a mean-convex smooth boundary ∂M. The main tool of this work is a suitable bridge principle for absolutely minimizing surfaces.

As is well known, minimal surfaces are in general not regular, not even if they are absolute minimizer. Geometric measure theory provides several objects to study minimizers and critical points even in the presence of singularities. In his talk Bob Hardt described a general theory of flat chains in metric spaces with coefficients in general groups. The theory developed by Hardt in collaboration with DePauw and Pfeffer unifies previous approaches by Brian White, Ambrosio and Kirchheim, which extended the classical theory of Federer and Fleming. A fundamental result in the regularity for area minimizing currents is Almgren's estimate of the size of the singular set, originally a preprint of 1728 pages. The understanding of Almgren's theory is a long standing issue and Emanuele Spadaro reported on the successful conclusion (jointly with Camillo De Lellis) of a program to reduce and simplify Almgren's proof, combining new methods in geometric measure theory with some new insightful ideas. Two other talks were strongly linked to the regularity theory of measure theoretic generalizations of minimal surfaces. The one of Nicola Fusco described a sharp quantitative version of Almgren's general isoperimetric inequality. Any closed m-dimensional surface Σ in $\mathbb{R}^N$ bounds a generalized surface (more precisely an integer rectifiable current) with volume at most $C(n)\,[\mathrm{Vol}^{\mathrm{m}}(\Sigma)]^{\frac{m+1}{m}}$. The optimal constant is achieved only by boundaries of flat $m+1$-dimensional disks. In a recent work with Verena Bögelein and Frank Duzaar, Fusco has studied, in a quantitative form, how a small deviation from the optimal constant forces the surface to be close to the boundary of a flat $m+1$-dimensional disk. A major tool in the regularity of minimal surfaces are excess decay estimates and corresponding regularity results for n-dimensional integral varifolds with generalized mean curvature in L^p. Ulrich Menne described in his talk a series of results where this question is investigated for the critical power $p = n$ and the subcritical powers $1 \leq p < n$, generalizing previous fundamental works by Allard and Brakke.

The Allen-Cahn equation has long been known to be strongly tied to the theory of minimal surfaces. Manuel Del Pino described in his talk the existence of entire solutions to the Allen-Cahn equation which are not one-dimensional and related results (joint work M. Kowalczyk, M. Musso, F. Pacard and J. Wei). The theorem of Del Pino and collaborators gives a counterexample, in high dimension, to a famous conjecture of De Giorgi.

Optimal transport is a rapidly-developing subfield of the calculus of variations that touches on a huge number of other areas, including geometry, physics, functional inequalities, applications in other areas such as economics, to name only some. The workshop featured three talks on these topics. A recent breakthrough in the regularity theory of the Monge-Ampère equation has been obtained by Guido De Philippis and Alessio Figalli. In his talk Figalli reported on their recent papers where they succeeded in proving Caffarelli's classical $W^{2,p}$ estimates without any smallness assumption on the L^∞ norm of the density, answering therefore a long-standing question in the field. In a first paper De Philippis and Figalli estimate the $L \log L$ norm of $D^2 u$. The tools of the paper were then used in one joint work with Ovidiu Savin to achieve L^p estimates: the same result has been proved independently by Thomas Schmidt, still relying on the first work by De Philippis and Figalli.

Other work on the Monge-Ampere equation included new regularity results for a class of optimal transportation problems arising in economics (by Young-Heon Kim, reporting on a joint work with Alessio Figalli and Robert McCann), as well as very recent work by Neil Trudinger in collaboration with Wei Zhang on Hessian measures in Heisenberg groups. The work of Trudinger and Zhang also resolves an outstanding problem for Heisenberg groups of dimension larger than two.

Geometric variational problems have provided the impetus for the development of much of the deepest theory in the calculus of variations, and this theory is continually being expanded, refined, clarified, while also finding new applications. Two talks focused on recent developments in the theory of Willmore surfaces, i.e. immersed surfaces Σ in Riemannian manifolds which are critical points of the Willmore functional $\int_\Sigma |A|^2 \, d\text{vol}$. Mondino surveyed several results obtained in collaboration with Kuwert, Rivière and Schygulla about the existence of Willmore surfaces in Riemannian 3-manifolds and of surfaces minimizing more general curvature functionals. The work of Kuwert, Mondino and Schygulla was described more in detail in the conference of Kuwert. Kuwert also reported on a theorem of Schygulla about the existence of minimizers of the Willmore energy in the class of 2-dimensional embedded surfaces in $\mathbb{R}^3$ having a fixed isoperimetric ratio.

A classical question in differential geometry concerns which smooth functions f can arise as Gauss curvature of a conformal metric on a 2-dimensional Riemannian manifold M. This amounts to solve a partial differential equation which is the Euler-Lagrange equation of an energy functional. Michael Struwe described a theorem with Franziska Borer and Luca Galimberti about the existence of a second critical point when the functional admits a strong minimizer. In their work the authors face the important difficulty that a Palais-Smale condition does not seem to hold for the relevant energy functional.

Guido De Philippis and Aldo Pratelli reported on recent progress in the study of variational problems involving eigenvalues of the Laplacian. In his talk Pratelli described the last available results and the most important open problems concerning shape minimizers of spectral problems, i.e. about domains Ω which minimize functionals of the eigenvalues of the Laplacian relative to suitable boundary conditions. Most of the talk focused on regularity properties of the minimizers, which were proved proved to exist by Buttazzo and Dal Maso in a suitable weak sense. In particular two theorems by Bucur and by Pratelli and Mazzoleni show the boundedness of the minimizers under very general assumptions. De Philippis reported on a joint work with Lorenzo Brasco and Bernardo Ruffini about the second eigenvalue of the Stekloff Laplacian. It is well known that the second eigenvalue of the Stekloff Laplacian achieves its maximum when the domain is a ball and the authors describe with a suitable inequality how much the domain is far from a ball if the second eigenvalue is close to such maximum.

Physics is a perennial source of problems in the calculus of variations. Talks this year on problem of physical origin were notably diverse, in terms of both the mathematical content and the physical models considered.

Deep new existence results for variational problems coming from gauge theory and high-energy physics have been proved recently using arguments developed over the past 10-12 years in foundational work on geometric measure theory in general metric spaces (see the talk of Tristan Rivière, joint work with Micea Petrache). Recent progress has been made on micromagnetics and related issues such as the Aviles-Giga functional through very thorough exploitation of entropy methods, as described in the talk by Ignat Radu.

Filip Rindler described new lower semicontinuity results for integral functionals connected to nonlinear elasticity which rely on new refinements, combining rigidity arguments with iterated blow-up constructions, of classical Young measure arguments. Stefan Müller reported on a rigorous proof of conjectural scaling laws for thin elastic films developing conical singularities (joint work with Heiner Olbermann).

Curvature-driven flows were addressed in the talks of Jörg Enders, Robert Haslhofer and Didier Smets. Enders described a joint work with Reto Müller and Peter Topping about blow-up points of type I for the Ricci flow, i.e. points p where the curvature tensor blows up at a rate $C(T-t)^{-1}$. Using the curvature control stemming from Perelman's pseudolocality theorem they show that the rescaled Ricci flow converges to a nontrivial soliton, thus answering positively a conjecture of Hamilton. The novelty is that in the rescaling of Enders, Müller and Topping the focal point is p itself and not some carefully chosen nearby point, as was customary in the previous literature. Haslhofer reported on a theorem obtained with Jeff Cheeger and Aaron Naber about the stratification of the singular set in Brakke's mean curvature flow. Cheeger, Haslhofer and Naber recover previous results by White and obtain new curvature estimates near the singular set exploiting a quantitative version of the usual strata subdivison in geometric measure theory.

Didier Smets described a new measure-theoretic approach to curves flowing by binormal curvature. In a joint work with Robert Jerrard the authors prove a global existence result, weak-strong uniqueness and some stability theorems for their generalized flow. Unlike the standard parametrized approach, which always yields smooth solutions, the flow defined by Jerrard and Smets captures the singularities which are motivated by the underlying physical models.

Workshop: Calculus of Variations

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