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Arbeitsgemeinschaft: Quasiperiodic Schrödinger Operators

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Abstract. This Arbeitsgemeinschaft discussed the spectral properties of quasi-periodic Schrödinger operators in one space dimension. After presenting background material on Schrödinger operators with dynamically defined potentials and some results about certain classes of dynamical systems, the recently developed global theory of analytic one-frequency potentials was discussed in detail. This was supplemented by presentations on an important special case, the almost Mathieu operator, and results showing phenomena exhibited outside the analytic category.

Mathematics Subject Classification (2000): 47B36, 81Q10.

Introduction by the Organisers

This Arbeitsgemeinschaft was organized by Artur Avila (Paris 7 and IMPA Rio de Janeiro), David Damanik (Rice), and Svetlana Jitomirskaya (UC Irvine) and held April 1–7, 2012. There were 31 participants in total, of whom 26 gave talks.

The objective was to discuss quasiperiodic Schrödinger operators of the form

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

with potential $V : \mathbb{Z} \to \mathbb{R}$ given by $V(n) = f(\omega + n\alpha)$, where $\omega, \alpha \in \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $\lambda \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ is such that $1, \alpha_1, \dots, \alpha_k$ are independent over the rational numbers, and $f : \mathbb{T}^k \to \mathbb{R}$ is assumed to be at least continuous.

H acts on the Hilbert space $\ell^2(\mathbb{Z})$ as a bounded self-adjoint operator. The associated unitary group $\{e^{-itH}\}_{t\in\mathbb{R}}$ describes the evolution of a quantum particle subjected to the quasiperiodic environment given by V. For any $t \in \mathbb{R}$ and $n \in \mathbb{Z}$, $|\langle \delta_n, e^{-itH}\psi \rangle|^2$ is the probability of finding the particle, whose initial state at time zero is given by the ℓ^2-normalized ψ, at time t at site n. The long-time behavior of these probabilities is of interest and many relevant questions about them can be studied by means of spectral theory (i.e., by "diagonalizing" the operator H). The spectral theorem for self-adjoint operators associates a measure μ_ψ with the initial state ψ. Roughly speaking, the more continuous the measure μ_ψ is, the faster $e^{-itH}\psi$ spreads. For this reason, one wants to determine the spectral type of H. For example, H is said to have purely absolutely continuous (resp., purely singular continuous or pure point) spectrum if every μ_ψ is purely absolutely continuous (resp., purely singular continuous or pure point). Again roughly speaking, the absolutely continuous case corresponds to transport, whereas the pure point case typically corresponds to the absence of transport ("dynamical localization"), while the singular continuous case corresponds to intermediate transport behavior. In the case where not all spectral measures have the same type, one collects all those states whose measures have the same type in a single subspace, restricts the operator to the resulting three subspaces, and the spectra of these three restrictions are then called the absolutely continuous, singular continuous, and pure point spectrum of H, respectively.

In the recent past, the spectral analysis of quasiperiodic Schrödinger operators has seen great advances. It was the goal of this Arbeitsgemeinschaft to present many of these advances.

It is useful to regard quasiperiodic potentials as being dynamically defined in the sense that they are obtained by sampling along the orbit of an ergodic transformation with a real-valued sampling function. Concretely, if we consider the map $T : \mathbb{T}^k \to \mathbb{T}^k$, $\omega \mapsto \omega + \alpha$, it is invertible and has normalized Lebesgue measure as its unique invariant Borel probability measure. Then, the potential V may be obtained as $V(n) = f(T^n \omega)$. More generally, whenever we have such a dynamically defined situation with an ergodic (Ω, T, μ) and a (bounded) measurable $f : \Omega \to \mathbb{R}$, several fundamental results hold. Namely, the spectrum, as well as the absolutely continuous spectrum, the singular continuous spectrum, and the point spectrum, of H are μ-almost surely independent of ω and are denoted by $\Sigma, \Sigma_{\rm ac}, \Sigma_{\rm sc}, \Sigma_{\rm pp}$. Moreover, the density of states dk, given by

$$\int_\Omega \langle \delta_0, g(H)\delta_0 \rangle \, d\mu(\omega) = \int_{\mathbb{R}} g(E) \, dk(E)$$

and the Lyapunov exponent

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \|A_E^n(\omega)\| \, d\mu(\omega),$$

where

$$A_E^n(\omega) = \begin{pmatrix} E - f(T^{n-1}\omega) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

are defined and determine Σ and $\Sigma_{\rm ac}$ as follows,

$$\Sigma = \operatorname{supp} dk, \qquad \Sigma_{\rm ac} = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{\rm ess}.$$

These two fundamental results can be considered classical and have been known since the early 1980's.

Note that $L(E) \geq 0$ and hence one naturally distinguishes between the two cases $L(E) = 0$ and $L(E) > 0$. The result quoted above shows that the first case is connected to the absolutely continuous part, whereas the general tendency is for the second case to be connected to the pure point part. Indeed, among the major recent advances are ways to go from positive Lyapunov exponents to pure point spectral measures (and more, such as exponentially decaying eigenfunctions, dynamical localization, etc.) in the quasiperiodic case with sufficiently regular sampling function f and for most α. Another major recent development is that in the case $k = 1$ (i.e., $\alpha, \omega \in \Omega = \mathbb{T}$), the regime of zero Lyapunov exponents has been studied in a global sense and it has been shown for analytic f, that one typically has purely absolutely continuous spectrum there. As a consequence, one now understands the typical spectral type of a one-frequency quasiperiodic Schrödinger operator with analytic sampling function.

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