

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Real Enumerative Questions in Complex and Tropical Geometry

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ABSTRACT. The workshop *Real Enumerative Questions in Complex and Tropical Geometry* was devoted to a wide discussion and exchange of ideas between the best experts representing various points of view on the subject. Enumeration of real curves largely motivated the development of the tropical geometry and led to the discovery of new interesting geometric phenomena and deep links between this problematic and algebraic geometry, symplectic geometry, topology, and mathematical physics.

Mathematics Subject Classification (2000): Primary 14-xx (algebraic geometry), Secondary 81Txx (quantum field theory, related classical field theories), 53Dxx (symplectic geometry, contact geometry).

Introduction by the Organisers

The workshop *Real Enumerative Questions in Complex and Tropical Geometry*, organized by Grigory Mikhalkin (Genève), Eugenii Shustin (Tel Aviv), Johannes Walcher (Genève), and Jean-Yves Welschinger (Lyon), was held April 18th–April 23rd, 2011. This meeting was well attended by with about 50 participants from around the world. The program of the workshop consisted of 18 one-hour talks given by leading experts in the subject as well as by perspective young researchers. In addition, four informal discussions on open problems and on questions related to the main topics of the workshop were ran during this week. Extended abstracts

of the talks and reports on the discussions follow these introductory notes. A special feature of the workshop was a dialog between mathematicians and physicists around the main subject.

The idea of the workshop was to put in the center the enumeration of real curves, which, on one side, appeared as a natural counterpart of the complex Gromov-Witten theory and which largely motivated the development of tropical geometry, and, on the other side, is deeply linked with counting pseudoholomorphic curves with Lagrangian boundary conditions in symplectic geometry and mathematical physics. Despite a number of striking results in this direction, serious problems remain open on the way to a systematic theory in real enumerative geometry. We shortly comment on these problems and on how they were reflected in the talks and discussions during the workshop.

About 15 years ago a huge breakthrough in Mathematics happened when Kontsevich suggested a way to enumerate complex rational curves in the framework of String Theory. Real algebraic geometry is almost always a much more delicate subject than its complex counterpart. Nevertheless, a significant progress in understanding real enumerative geometry was done recently. A breakthrough was provided by the discovery by J.-Y. Welschinger (one of the organizers) of a way to invariantly enumerate rational curves in two and three dimensions passing through point constraints. In the same time a technique of tropical enumeration was developed (G. Mikhalkin). It allows to enumerate real and complex curves simultaneously which eventually amounts in the enumeration of certain graphs matching given constraints and equipped with “complex” or “real” combinatorial weights. Among arising key problems we mention (i) the understanding of real tropical enumerative invariants and related “cycles” in moduli spaces of tropical curves, (ii) The lack of appropriate correspondence theorems for the Kontsevich WDVV equation, computation of ψ -classes and some other problems of the complex enumeration, which could indicate their possible real enumerative analogues, (iii) the search for real enumerative invariants matching recently discovered real tropical invariants of positive genus. Particularly, the last problem reduces to a rigorous definition of the correction term to the Welschinger count, which is one of the main problem of the area. Considerations coming from Physics support the approach that includes enumeration of both type I and type II curves. In turn, enumeration of type II curves lies outside of a well-established Symplectic geometry approach of enumerating holomorphic curves with boundary on Lagrangian submanifolds (open Gromov-Witten theory). From the physical point of view, the real and complex enumerative geometry appears in topological quantum field theories and topological string theory, mirror symmetry and open Gromov-Witten theory.

Four survey lectures opened the workshop and presented the state of the art in the topological aspects of real and complex enumerative geometry (O. Viro), Lagrangian Floer theory as a symplectic side of the story (K. Ono), (p, q) -branes in string theory with relations to tropical and enumerative geometry (A. Hanany),

recent developments in the tropical enumerative geometry (I. Itenberg). The tropical theory has been in the focus of the talks by E. Brugallé, A. Gathmann, K. Shaw, I. Tyomkin. Among them we especially mention the contribution by E. Brugallé, where the use of tropical modifications allowed to extend the tropical techniques beyond the range of toric examples, and the results presented by A. Gathmann, which resolve the lack of local non-invariance of tropical Welschinger invariants for configurations with imaginary points. The latter talk surprisingly resembles the approach to a correct definition of relative open Gromov-Witten invariants presented by R. Rasdeaconu. A common idea, which potentially can shed light on the problem of counting real curves of higher genus (cf. the discussion led by G. Mikhalkin), is to combine enumeration of real, resp. tropical curves of different kinds which together constitute an invariant. Very interesting topological ideas in the enumerative geometry have appeared in the talk by M. Polyak and in the informal discussion led by M. Kazarian. The promising picture of physical predictions in counting real holomorphic curves in Calabi-Yau three-folds was developed in the talk by D. Krefl, which also presented a physical intuition behind the invariant count of real curves. The lectures by N. C. Leung and K. Fukaya were devoted to the mirror symmetry of toric Calabi-Yau varieties, where the counting of signed (pseudo)holomorphic discs with suitable Lagrangian boundary conditions naturally enters the story. In its turn, the talk by M. Mariño linked the mirror symmetry of toric Calabi-Yau with spectral curves and their tropical limits, which opens a perspective application of tropical geometry. The algebra-geometric theory of mirror symmetry was discussed in the talk by L. Katzarkov. The talk by Y.-H. He exposed a wide physical picture involving quantum field theories, quiver gauge theories, dimer models, leading to amoebas and the tropical curves as their limits. The superpotentials describing interactions in the considered there field theories appeared a subject of a special discussion led by J. Walcher. In its turn, the complex-analytic point of view on amoebas and co-amoebas showed up in the discussion led by M. Passare. The symplectic problematic in the lecture by V. Shevchishin shares techniques and ideas with main problems of the workshop.

We believe that a very intensive and substantial exchange of a broad spectrum of ideas during the workshop will really stimulate a further research in the main discussed problems, which still are far from being completely settled.

