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Calculus of Variations

Organised by
Camillo De Lellis, Zürich
Gerhard Huisken, Golm
Robert J. McCann, Toronto

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Abstract. Since its invention by Newton, the calculus of variations has formed one of the central techniques for studying problems in geometry, physics, and partial differential equations. This trend continues even today. On the one hand, slow but steady progress is made on long-standing questions concerning minimal surfaces, curvature flows, and related geometric objects. Basic questions also remain in such areas as mathematical physics and general relativity. On the other hand, new types of question emerge, driven by applications from economics and engineering to materials science, whose solution will depend on developing ideas and techniques in this classical branch of analysis. The July 2010 Oberwolfach workshop on the Calculus of Variations showcased a blend of continued progress in traditional areas with surprising developments which emerged from the exploration of new lines of research.

Mathematics Subject Classification (2000): 49-xx, 35Jxx, 53Cxx, 58Exx.

Introduction by the Organisers

This workshop attracted 49 participants, including 13 recent PhDs and 3 women. Its main themes could be divided into four large groups (i) geometry (ii) partial differential equations; (iii) physics and materials; (iv) optimal transportation and its applications. Doctoral students and postdoctoral fellows accounted for nearly a third of the 21 presentations which took place 19-23 July 2010.

The first general area encompassed the role of calculus of variations in differential geometry, including minimal surface theory and general relativity. Some of the most exciting developments here concern rigidity questions in Riemannian geometry described by Simon Brendle and Andre Neves. Brendle's lecture was devoted to the construction of a counterexample to a conjecture of Min-Oo. For a manifold which is asymptotically Euclidean and has non-negative scalar curvature, the positive mass theorem asserts that the ADM mass be non-negative, vanishing only in the case of Euclidean space. Min-Oo established a similar result in the asymptotically hyperbolic setting — namely, that no compact perturbation exists whose only effect is to increase the positive scalar curvature locally. He conjectured the same would be true in the positive curvature setting of the hemisphere, a conjecture disproved by Brendle's example with Marques and Neves. Neves, on the other hand, devoted his talk to positive results, including sharp bounds on the area of a minimizing surface in a compact oriented 3-manifold whose scalar curvature exceeds that of the sphere; the case of equality is attained by products of spheres.

Curvature-driven flows were addressed in talks by Peter Topping, John Head and Brian White. Peter Topping explained conditions for there to be a unique Ricci flow which instantaneously completes an incomplete surface. The talks of Head and White concerned flows of embedded submanifolds by mean curvature. Here a program by Huisken and Sinestari has succeeded in classifying singularities of the flow; as in the Ricci flow case, such singularities can be bypassed using surgery. John Head described doctoral work showing that in the limit, the flow obtained by postponing the surgeries for as long as possible coincides with the one arising from the viscosity solution of the level-set formulation of mean-curvature flow. On the other hand, Brian White explained how he and Tom Ilmanen have exploited the classification of singularities for mean-curvature flow to resolve a classical problem in geometric measure theory. Under a mild topological assumption, they were able to show that the density of an area-minimizing hypersurface exceeds the square root of two at each singularity. Among dimension independent bounds, this result is sharp. Emanuel Spadaro described his doctoral work with Camillo DeLellis, which focused on simplifying Almgren's thousand page proof of regularity results for minimal varieties of codimension two and higher. Finally, Mu-Tao Wang described his definition with S.T. Yau of the quasi-local mass (or total energy) bounded by a closed spacelike surface in general relativity. Their approach, which involves extremizing over isometric embeddings of the geometry into a Lorentzian spacetime, is reminiscent of Gromov's definition of the Hausdorff distance between two abstract metric spaces.

Turning to variational problems in physics and materials science, we may mention the review of Felix Otto, devoted to establishing ansatz-free upper bounds on nonlinear rates of coarsening in dynamical settings, and on branching formations in static patterns. Here interpolation inequalities between function spaces sensitive to competing energies in the physical system play crucially. Robert Seiringer described classical and quantum mechanical models for electrons moving through a dielectric medium, and explained how screening effects must be taken into account when analyzing the ground state energy of the system, to preclude the possibility of binding. Nicola Fusco discussed the variational problems governing the equilibrium configurations of an epitaxially strained crystalline film.

Exciting developments were also reported in the theory of partial differential equations which arise as Euler-Lagrange equations for variational problems. One of the highlights was Neil Trudinger's description of affine maximal hypersurfaces. This geometric problem dates back to Chern and Calabi, and involves proving regularity for 4th order analogs of the elliptic Monge-Ampère equation. For two-dimensional surfaces, the problem was solved some fifteen years ago by Trudinger and Wang. In recent work, they have succeeded in extending their result to all dimensions. Giuseppe Mingione described new techniques for showing boundary regularity of solutions to minimization problems below the level at which the Euler-Lagrange equation becomes effective; the key technical problem is that coefficients in this equation depend on the solution, hence need not be smooth, a priori. Alessio Figalli described regularity results with Luis Caffarelli for an obstacle problem from mathematical finance involving the fractional Laplacian, while Daniel Faraco described the lack of uniqueness for solutions of the incompressible porous medium equation, following similar results in fluid mechanics dating back to Shnirelman. Yet another highlight was new PhD Charles Smart's lecture on optimal Lipschitz extensions. Here he described the differentiability proved with Lawrence C Evans for the viscosity solution of the infinity-Laplace equation, and his simplification with Armstrong of Jensen's argument for its uniqueness. He concluded by describing preliminary results concerning the vector-valued analog of this problem, which is to construct a mapping whose Lipschitz constant is the minimum possible (relative to its boundary conditions) on every subdomain of a given domain.

Turning to questions in optimal transportation, Brendan Pass described doctoral research on the multiple marginal problem of optimally correlating $m \geq 3$ distributions in several dimensions with respect to a given cost function. He described existence, uniqueness, and rectifiability results, some of which were new even for two marginals. Most striking among these is difference between the dimension of the maximizer and the minimizer when $m \geq 3$, and the fact that the solution is a spacelike manifold with respect to 2^{m-1} pseudo-metrics. Young-Heon Kim described progress with Figalli and McCann concerning regularity of optimal maps on a Riemannian manifold in the two marginal case. They have overcome the subtleties associated with the cut-locus on products of round spheres, which provide a reasonably robust model for the singularities displayed by more general superdifferentiable costs. Alexander Plakhov described problems of minimizing aerodynamic resistance, in which optimal transportation plays a role. His delicate constructions establish the surprising ability to lower the resistance to zero in certain (unstable) directions. To do so requires recapturing lost momentum through multiple scattering. Finally, Guillaume Carlier described an economic model for optimal transportation with congestion.

Apart from the lectures classified above, there were several which defy categorization, such as Paul Lee's results on bracket conditions which guarantee the continuity of sub-Riemannian actions in a control-theoretic context, and Bob Jerrard's talk on Lorentzian analogues of variational questions modeling the limiting geometry of singularities arising from phase-field models in a singular limit. In addition to formal lectures, many lively discussions between new and seasoned researchers took place throughout the week, affirming the vitality of this flourishing subject. We hope the collection of extended abstracts supplied by the speakers below helps to convey a sense of the excitement and possibilities shared by the participants and researchers working at the scientific frontier in the calculus of variations.