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Motives and Homotopy Theory of Schemes

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Abstract. The 2010 program on Motives and Homotopy Theory of Schemes consisted of a lively and varied series of 19 one-hour lectures on the latest developments in the field, presenting a wide range of aspects of this multi-faceted subject. Besides the lectures, we were happy to welcome a wide range of nationalities and age groups to the conference.

Mathematics Subject Classification (2000): 14F, 14G, 19E.

Introduction by the Organisers

The confluence of algebraic geometry and homological algebra known as the theory of motives has experienced an amazing resurgence of activity in the last twenty years. More recently, the growth of motivic homotopy theory has expanded the area to allow for a systematic treatment of a wide variety of “motivic” phenomena, embedding $K$-theory, motivic cohomology, quadratic forms into a single larger field. At the same time, the theory allows for the transfer of constructions and techniques from classical homotopy theory to problems in algebraic geometry.

Here in more detail are the topics which were discussed.

Motives, varieties and algebra. We had three talks on applications of motives to the study of varieties over non-algebraically closed fields. Using a version of the Rost motive, Semenov described a surprising restriction on the Rost invariant for homogeneous spaces for $E_6$, Gille extended the property of Rost nilpotence to geometrically rational surfaces over fields of characteristic zero, Zainoulline gave
a uniform bound for the torsion part of codimension 2 algebraic cycles on certain projective homogeneous varieties. In addition, Krashen explained how the patching techniques of Harbater and Hartmann were applied (in a joint work with these two) to gives a new local-global principles for galois cohomology.

**Categories of motives.** Déglise described his work with Cisinski constructing a category of motives (with \( \mathbb{Q} \)-coefficients) over a general base satisfying the Grothendieck six operations formalism. Barbieri-Viale showed how Nori’s construction of a category of motives gives a finer construction of a category of \( n \)-motives, i.e., motives of varieties of dimension \( \leq n \), with \( n = 0 \) being the category of Artin motives, \( n = 1 \) Deligne’s category of 1-motives. Park described his construction (with Krishna) of a triangulated category of motives over \( k[t]/t^{n+1} \), based on modifications of the Bloch-Esnault additive Chow groups. Wildeshaus showed how he applied the technique of weight structures on a triangulated category, developed by Bondarko, to study motives of Shimura varieties.

**Tannaka groups and fundamental groups.** Esnault described her proof (with Mehta) of Gieseker’s conjecture, that the vanishing of the étale fundamental group of a smooth projective variety \( X \) over an algebraically closed field of positive characteristic implies that there are no non-trivial \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules on \( X \). Terasoma described his construction (with K. Kimura) of a mixed cycle-theoretic and representation-theoretic differential graded algebra, whose co-modules may be viewed as “mixed elliptic motives”. Furusho described his work giving simplified relations defining the Grothendieck-Teichmüller group, and showing that all elements of the Grothendieck-Teichmüller group satisfy the “double-shuffle relations”.

**Arithmetic.** Geisser discussed Parshin’s conjecture, that the rational higher \( K \)-theory of a smooth and proper variety over a finite field is torsion, and related this conjecture to finite generation properties of motivic cohomology and motivic Borel-Moore homology, as well as the statement that rational motivic homology and cohomology are dual vector spaces. Flach reported on progress (including joint work with Morin) in Lichtenbaum’s program of describing the vanishing order and leading term of zeta functions of arithmetic schemes in terms of Weil-étale cohomology. In particular, Flach and Morin have defined a Weil-étale topos for a regular proper scheme over \( \text{Spec} \mathbb{Z} \) which gives the correct answer for the zeta value at 0. Holmstrom reported on his work (with J. Scholbach) on lifting the Deligne regulator to a map in the motivic stable homotopy category, and using this to define Arakelov motivic cohomology via a cone construction.

**Motivic homotopy theory.** Ostvar discussed his computations (with Ormsby) giving information on the coefficient rings for \( MGL, kgl \) and the motivic sphere spectrum, using versions of the Adams spectral sequence and the Adams-Novikov spectral sequence. Pelaez presented his recent work on the functoriality of the
slice filtration, which as an application gives a good definition of an integral category of motives over a base-scheme \( S \) for \( S \) a scheme over a field of characteristic zero. Yagunov showed us his computation of the first non-trivial differential in the motivic cohomology to \( K \)-theory spectral sequence, after localization at a given prime. His main result is that this differential is expressible in terms of the motivic Steenrod operations. Asok reported on a joint work with Morel and Haesemeyer, in which they compute the maps in the motivic stable homotopy category from \( \text{Spec} \ k \) to a smooth proper scheme \( X \) as the group of oriented 0-cycles on \( X \) (as defined by Barge-Morel and extended by Fasel). Hornbostel gave us a description of a motivic version of a result of Lurie in the stable homotopy category, namely, that the suspension spectrum of \( \mathbb{CP}^\infty \) classifies “preorientations of the derived multiplicative group”. This motivic version gives as an application an intrinsic description of algebraic \( K \)-theory, namely, that it represents orientations of the derived motivic multiplicative group.