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# Moduli Spaces in Algebraic Geometry

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Abstract. The workshop on Moduli Spaces in Algebraic Geometry aimed to bring together researchers from all branches of moduli theory, in order to discuss moduli spaces from different points of view, and to give an overview of methods used in their respective fields. Highlights included a complete proof of Göttsche's conjecture, a proof of rationality of a moduli space constructed via GIT quotient using reduction modulo $p$, and a proof of a conjecture of Looijenga using the ideas of mirror symmetry.

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## Introduction by the Organisers

The workshop *Moduli Spaces in Algebraic Geometry*, organised by Dan Abramovich (Brown), Gavril Farkas (HU Berlin), and Stefan Kebekus (Freiburg) was held January 10–14, 2010 and was attended by 52 participants from around the world. The participants ranged from senior leaders in the field to young post-doctoral fellows and even a few PhD students; their range of expertise covered areas from classical algebraic geometry to motivic Hall algebras.

Being central to a number of mathematical disciplines, moduli spaces are studied from many points of view, using a wide array of methods. Major progress has been achieved in virtually every branch of the field, and well-known questions have been answered lately. The workshop brought together researchers working on different aspects of moduli theory, to report on progress, discuss open problems, give overview, and in order to exchange methods and ideas. Lecture topics were chosen to cover many of the subject's disparate aspects, and most lectures were followed by lively discussions among participants, at times continuing well into the night.

For a flavor of the wide palate of subjects covered, a few of the talks are highlighted below.

**Characteristic classes on surfaces. Proof of Göttsche's conjecture.** Jun Li (Stanford) lectured on the solution of Göttsche's conjecture, obtained by his student Yu-jong Tzeng. Although the result was announced a few months ago, his talk in this workshop was the first time a complete proof was presented in a public lecture in Europe.

Given an algebraic surface $X$ and a suitably general $m$-dimensional linear system $V$ on $X$, the problem of counting the number of $m$-nodal elements of $V$ can be traced back to the 19th century. Göttsche's conjecture predicts that, in a suitable range, this number is a universal function of four characteristic classes of $X$ and $V$. Göttsche reduced his conjecture to a statement on intersection numbers on $Hilb(X)$. The key ideas in the proof of Tzeng are (a) a spectacular generalization of the work of Levine and Pandharipande on generators of the cobordism group of pairs $(X, L)$ of a surface with line bundle, and (b) an equally spectacular proof of a degeneration formula showing that Göttsche's intersection numbers are cobordism invariants.

**Understanding deformations using mirror symmetry.** Paul Hacking (Amherst) presented a solution to a 28-year-old conjecture of Looijenga, using ideas that originate from mirror symmetry.

A surface cusp singularity has a cycle of rational curves as its exceptional configuration. In 1981 Looijenga conjectured that a cusp singularity is smoothable if and only if the exceptional set of the *dual* cusp lies on a rational surface as an anticanonical divisor. Gross, Hacking and Keel later recognized the appearance of the configuration on a rational surface as part of a construction coming from mirror symmetry. In this setting, mirror symmetry works perfectly: counting rational curves on the mirror dual, one obtains an explicit deformation of a given cusp. Looijenga's conjecture follows.

**Topology of moduli spaces and their relative connectivity.** Eduard Looijenga (Utrecht) discussed topological properties of moduli spaces, presenting a result of a very classical flavor. He reported on joint work with W. van der Kallen, proving the vanishing of the relative homology groups $H_k(\mathcal{A}_g, \mathcal{A}_{g,dec}; \mathbb{Q})$ for $k \leq g-2$, where $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties and $\mathcal{A}_{g,dec}$ is the locus of decomposable ones. The proof goes by a sequence of beautiful reductions, proving in particular that the corresponding decomposable locus on the Siegel space is homotopy equivalent to a bouquet of $(g-2)$-spheres, which in itself is reduced to a completely combinatorial problem.

An analogous result holds for moduli of curves of compact type: we have $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ for $k \leq g-2$. Here, the result follows from a combinatorial discussion of the separating curve complex, which is shown to be $(g-3)$-connected.

**Using computer algebra to prove rationality.** Christian Böhning (Göttingen) reported on his joint work with H.-C. von Bothmer on rationality of the space of plane curves of sufficiently high degree $d \geq 0$. The starting point of the proof is rather classical and uses the Aronhold method of covariants which gives a map from the space of degree $d$ plane curves to that of quartic curves. In order to show that a general fibre of this map is a vector bundle over a rational base, a certain matrix having entries polynomials in $d$, must have full rank. To achieve this, the authors introduce innovative techniques that rely on reduction to characteristic $p$ and a computer check of the corresponding statement over a finite field. By semicontinuity, then rationality follows in characteristic 0 as well!

**Tautological rings of the moduli space of curves.** Carel Faber (Stockholm) gave the inaugural talk of the workshop and discussed developments about certain subrings of the cohomology of the moduli space $M_g$ of curves of genus $g$. Around 1993, Faber formulated an amazing conjecture predicting that the tautological ring of the $(3g-3)$-dimensional moduli space $M_g$ enjoys all the properties (vanishing, perfect pairing), of a smooth compact complex manifold of dimension $g-2$. Faber's Conjecture generated a great deal of interest in the last few years, and significant parts of it (vanishing, top degree predictions) have been confirmed. However, the part predicting the existence of a perfect pairing between complementary tautological rings has been more resistant to proofs. Quite surprisingly, it turns out that the Faber-Zagier method of producing enough tautological relations to verify this part of the conjecture, stops working exactly in genus 24! The occurrence of this genus in relation to Faber's Conjecture has caused quite a stir, especially since this is also the range when $M_g$ starts to become a variety of general type.