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Infinite Dimensional Random Dynamical Systems and Their Applications

Organised by
Franco Flandoli, Pisa
Peter E. Kloeden, Frankfurt am Main
Andrew Stuart, Coventry

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ABSTRACT. The theory of infinite dimensional random dynamical systems shares many of the concepts and results for finite dimensional random dynamical systems, but with considerably more technical complications. Many examples are generated by stochastic partial differential equations (SPDE) which are used to model climate dynamics, turbulence, porous media, random surface motions and many other systems of physical interest, etc. The workshop covered broad spectrum of issues ranging from the theoretical to applications and numerics.

Mathematics Subject Classification (2000): 37Hxx, 37Lxx, 60Kxx.

Introduction by the Organisers

The theory of random dynamical systems presented in the monograph of Ludwig Arnold (*Random Dynamical Systems*, Springer-Verlag, 1998) has given rise to completely new understanding of the dynamics of stochastic differential equations and has been particularly successful for finite dimensional systems when applied to specific examples. Many of the ideas and results can be generalized to infinite dimensional random dynamical systems. Here the theory meets the field of stochastic partial differential equations (SPDE) which is an active area of recent research with strong connections with relevant problems of physics. SPDEs are used to model climate dynamics, turbulence, porous media, random surface motions and many other systems of physical interest. Asymptotic properties analyzed by the theory of random dynamical systems like invariant measures, attractors, stability, are of clear importance for the understanding of the long time behavior of these systems. Numerical simulation, system reduction and parameter estimations are other related topics of interest.

The aim of this workshop was to gather experts of SPDE and random dynamical systems, some of them more physically or numerically oriented, other with more theoretical experience, in order to exchange their ideas about the state of the art and the main open problems they investigate.

1. Structure and style of the talks

We had 25 one-hour lectures. Most of the participants gave a talk and three speakers gave very short series of talks.

The short series, devoted to Lévy noise driven equations (Zabczyk), stochastic porous media equations (Röckner), stabilization by noise and random attractors (Crauel) have been devised as an introduction to new emerging research directions or fundamental topics of general interest for the participants.

All the other single talks were allowed to be one-hour long in order to offer each speaker the possibility to introduce more smoothly the subjects and give more details. The result of this choice, as generally acknowledged by the participants, has been that it was possible to learn more than usual from the talks and ask freely a lot of questions. Having a good 10 minutes of time at the end of each talk meant that an unusually large number of questions could be asked and the speakers had adequate time to answer them with the necessary details.

2. Topics of the lectures

The full list of talks is reported below in detail, so here we limit ourselves to group them following different conceptual lines, with some repetition.

Applications in Physics. Analysis of models and questions of interest for Physics has been part of the presentations of several authors. *Röckner* devoted his lectures to porous media equations and proved, for instance, an extinction property as an example of self-organized criticality. *Kuksin* considered KdV equations perturbed by noise and the zero-noise limit problem, proved an averaging result for action/angle variables and identified the physical invariant measure of the deterministic KdV equation. *Kotelenez* revised Einstein theory of Brownian motion in the case of two or more large particles suspended in a medium and quantified their correlation at short distance, justifying in this way the use of correlated noise in many examples of SPDEs. *Kondratiev* considered a stochastic birth and death system of infinitely many particles in the continuum and described a general mathematical framework for their analysis. *Duan* considered SPDEs related to geophysical models, in the framework of the problem of model uncertainty, and described a way to estimate Hurst parameter and coefficients of the noise to fit data as best as possible. *Imkeller*, again in the field of climate modelling, considered temperature data of the last millions of years and compared the performances of equations based on Levy and Brownian noise to produce qualitative features similar to those of these data. Sharp interface random motion described by a stochastic version of Allen-Cahn equation have been presented by *Romito* who analyzed birth and annihilation of random interfaces.

Porus media. As we already mentioned, porous media equations received particular attention. The talks of *Röckner* ranged from the foundational results expressed in very large generality, to the analysis of special properties like the extinction and the existence of random attractors. *Russo* gave a probabilistic representation formula for solutions of the deterministic porous media equations in very singular situations by means of a nonlinear Fokker-Planck equation, which gives particular insight in the structure and behavior of solutions and provides new numerical methods. *Wei Liu* proved dimension-free Harnack inequalities in the sense of Wang for the stochastic porous media equations and obtained as a consequence regularity of transition kernel and ergodicity.

Lévy, fractional Brownian and other noise perturbations. While the main body of past and present investigation in SPDE is concerned with Brownian noise, several recent investigations are concerned with Lévy noise and fractional Brownian noise. *Zabczyk* gave two introductory talks on infinite dimensional stochastic equations driven by Lévy noise, providing both foundational results of the properties of infinite dimensional Lévy processes and advanced results of time-regularity, support and regularity of the law for the solutions to the equations. *Imkeller* considered equations with multiple-well potentials driven by α-stable processes, estimated the probability of tunnelling and described the relation between this phenomenon and observations of temperature changes in climate data. Fractional Brownian motion entered the SPDEs of Schmalfuss and Duan. *Schmalfuss* described foundational results on infinite dimensional equations driven by fractional Brownian motion and proved a result of existence of local unstable manifold. *Duan* showed how to detect the parameters of fractional Brownian noise to fit experimental data. Both the works of Duan and Imkeller show that interest of these noise terms in real applications. *Gubinelli* gave an introduction to the less traditional formulation of differential equations driven by rough paths, both in finite and infinite dimensions, which covers a large class of fractional Brownian motions in particular, and reviewed recent results on SPDE including the tree expansion approach to nonlinear problems.

Numerics. Numerical analysis and simulation of SPDE is a rather recent and very important subject for the future of the field and its applications. *Kloeden* reviewed recent results about error estimates and showed particular improvements of more classical results. Simulations of stochastic Burgers type equations have been showed by *Blömker* to illustrate the stabilization by noise associated to amplitude equations, described below. *Romito* described the problems emerging in the numerical approximation of sharp interface stochastic equations due to the presence of steep gradients and space-time white noise.

Stabilization by noise. Noise is usually associated to the idea of perturbation and destabilization, but in certain circumstances it may have a stabilizing effect. *Crauel* showed how noise can stabilize an unstable deterministic system by means of rotations which average stable and unstable directions, applied this technique to a nonlinear stochastic parabolic equation and showed other stabilizability results by random and non random sources. *Blömker* proved by means of amplitude

equations that an additive noise in the second Fourier component of the stochastic Burgers equation may have the effect of a multiplicative noise acting on the first component and stabilizes it.

Stochastic flows. Stochastic flows are at the core of the theory of random dynamical systems: they are the random analog of deterministic flows and allow to analyze properties which depend on the simultaneous action of the dynamic on different initial conditions. *Le Jan* considered stochastic flows associated to isotropic Brownian motion, both in the regular case of flows of maps and in the generalized case of random Markov kernels, and proved a pathwise central limit theorem with respect to randomness in the initial conditions. *Scheutzow* considered the problem of existence and possible non existence of stochastic flows for stochastic equations which have unique global solutions for each individual initial condition and gave examples of non existence of flows in certain classes of regularity of coefficients.

Random and deterministic attractors and invariant manifolds. The existence and the structure of attractors and other invariant sets are some of the most basic questions to ask for dynamical systems. Several talks mentioned this topic. In particular, *Crauel* revised the concepts of strong and weak random attractor, their main properties and presented new sufficient conditions for their existence based on probabilistic estimates. *Schmalfuss* applied a stochastic version of Lyapunov-Perron transformation to construct local unstable manifold for SPDE driven by fractional Brownian motion. *Brzeźniak* proved the existence of a compact random attractor for the 2D stochastic Navier-Stokes equations in unbounded domain, where classical arguments of compactness do not apply. *Maier-Paape* considered Cahn-Hilliard equations or more generally gradient type systems and gave elements of the theory of Conley index theory for the purpose of resolving the fine structure of invariant sets.

Finally, *Hairer* presented a full picture of classical ergodic theory and the recent improvements based on the concept of asymptotic Strong Feller property, with application to a general class of SPDE which includes 2D stochastic Navier-Stokes equations, while *Johnson* described the recent theory of Sturm-Liouville problems with algebro-geometric potentials, related to solutions of the Camassa-Holm equation and with analogies with the theory of Schrödinger and KdV equations.