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Geometry and Arithmetic around Hypergeometric Functions

Organised by
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Abstract. Hypergeometric functions form a classical subject of Mathematics providing an interesting link between apparently quite different fields of research like differential equations, group theory, differential geometry, physics, computational mathematics, moduli spaces, arithmetic and algebraic geometry. The conference succeeded to give insights into new developements in many different directions and to encourage discussions and exchange of ideas between participants who would never had met otherwise or at other places than Oberwolfach.

Mathematics Subject Classification (2000): 33C, 11F, 11J, 14G, 14M.

Introduction by the Organisers

The workshop *Geometry and Arithmetic around Hypergeometric Functions*, organised by Gert Heckman (Nijmegen), Jürgen Wolfart (Frankfurt a.M.) and Masaaki Yoshida (Fukuoka) was held September 28th – October 4th, 2008. It was attended by more than 40 participants coming from more than 10 different countries and representing a remarkable variety of mathematical research directions centered around hypergeometric functions. The public of the conference was mixed not only with respect to its mathematical interests but also in age and experience: it ranged from doctoral students to excellent and well–known senior researchers.
The program concentrated on important new developements of the last years, let sufficient time for discussions in between the talks, and gave also promising younger people the possibility to present their work, partly in more informal evening seminars. Even the contributions given on Friday afternoon and evening were attended

by almost all participants, so we may assume that the program was greatly appreciated.

A long standing question about algebraicity and transcendence of special values of Gauss hypergeometric functions has been answered in recent years using work of Wüstholz, Edixhoven, Yafaev and Klingler about conjectures concerning special subvarieties of Shimura varieties. The conference had a special day with four featured talks of these authors about the subject and an informal evening seminar concerning generalizations of this question to higher dimensions.

A remarkable event on the classical number theoretic side of hypergeometric functions was Don Zagier's featured talk explaining a new link of complex differential equations to Hilbert and Teichmüller modular forms.

The arithmetic–geometric mean can be written in terms of the hypergeometric function – a classical fact. Matsumoto-Shiga reported about possibilities of its genralization.

Algebro–geometric aspects of the Painlevé equations and their generalizations were reported by van der Put and Tsuda. It is well known that these equations are related to the hypergeometric equations and its generalizations.

About the classical hypergeometric equation, Vidunas proved that there are still many things to be studied, and Sasaki–Yoshida presented differential geometric and singularity theoretic aspects of the hyperbolic Schwarz map.

The deep relation between hypergeometric differential equations and ball quotients originated with Schwarz, Picard, Terada, and Deligne-Mostow. New compactification techniques of such ball quotients (and more generally of so called Heegner divisor complements in such ball quotients) were reported by Heckman. These compactifications were introduced in full generality by Looijenga, inspired by the work of Heckman and Looijenga on the moduli space of rational elliptic surfaces. The geometric examples discussed in the lecture of Heckman were all related to groups generated by order four complex reflections (also called tetraflections) and go back to the work of Deligne-Mostow and of Kondo. Allcock presented a surprising conjecture about a ball quotient and the monster group. In this lecture the complex reflection groups were related to order three complex reflections (triflections in Conway's terminology), and in a sense the lecture by Heckman also served as a kind of introduction for Allcock's lecture. The conjecture by Allcock is truely amazing, and would in fact reveal a natural geometric surrounding (via generalized hypergeometric periods) for most of the sporadic groups. Holzapfel and his collaborators discussed towers of ball quotients and morphsims among these.

A vast generalization of the hypergeometric equation developed during the last twenty years is the GKZ (Gelfand–Kapranov–Zelevinsky) equation was another important subject of the meeting. Beukers presented a criterion for their solvability by algebraic functions, and Alicia Dickenstein reported about examples admitting rational solutions.