

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Mini-Workshop: Complex Approximation and Universality

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ABSTRACT. The mini-workshop was devoted to the study of universality in connection with approximation theory in complex analysis and related notions within the framework of Banach spaces and various function spaces. Topics discussed included invariance of universality, Baire category and universality, the size and structure of the set of universal functions, universality for composition with respect to a sequence of mappings, universal Taylor series and Faber series, overconvergent series, lacunary series, holomorphic continuation of series, universal series with restricted coefficients, universality for solutions of the Laplace equation, the heat equation and Burgers' equation, hypercyclicity, and superhypercyclicity. Topics in approximation included polynomial approximation, approximation and maximum principles, approximating inner functions by Blaschke products, and approximation of the Riemann zeta function. A list of open problems is included in the report.

Mathematics Subject Classification (2000): 30Bxx, 30E10.

Introduction by the Organisers

The notion of universality covers a wide range of phenomena in complex analysis. Generally speaking, a universal object is one which, when subjected to some limiting process, approximates every object in some universe. For example, universality occurs when the translates of an entire function can approximate any other entire function, or when the partial sums of a formal power series or a formal trigonometric series approximate all functions in some natural class. For a long time, existing approximation theorems were used in constructions of universal functions and universal series. In recent years, however, constructions have required the development of new approximation theorems, thereby also enriching the area of complex approximation.

Universal functions. There is no single definition of a universal function. What they have in common is the following. One considers a suitable sequence $\mathcal{T} = (T_n)$ of operators acting on a space X , for example, of holomorphic functions with values in another space Y of holomorphic functions. Then a function $f \in X$ is called universal with respect to \mathcal{T} if the sequence $(T_n f)$ is dense in Y . One of the earliest examples of a universal function is due to Birkhoff (1929) who showed that there exists an entire function f whose translates $f(z+n), n \geq 1$, can approximate any other entire function, uniformly on compact sets. In that case we have $(T_n f)(z) = f(z+n)$, and $X = Y$ is the space of entire functions with the usual compact-open topology.

Seidel and Walsh showed that an analogue of Birkhoff's universality theorem holds for functions holomorphic in the unit disc, if we replace translates by "non-euclidian translates", that is $T_n f = f \circ \phi_n$ is the composition of f with an automorphism ϕ_n of the unit disc D . At the heart of the study of holomorphic functions in the disc D is the class $H^\infty(D)$ of bounded holomorphic functions on the disc. Chee showed the existence of universal functions for the class $H^\infty(B)$ of bounded holomorphic functions on the unit ball of C^N . Richard Aron's talk was concerned with the size and the structure of the set of such universal functions. In the study of the space $H^\infty(B)$ a fundamental role is played by inner functions. These are also of importance in engineering control theory. Recently, Gauthier and Xiao have shown the existence of universal inner functions in the unit ball of C^N . Geir Arne Hjelle and Raymond Mortini gave talks concerned with approximating inner functions in the unit disc D by simpler inner functions, namely Blaschke products.

Extending the study of functions in the unit disc, which are universal with respect to composition with automorphisms of the disc, Mortini talked about the universality of functions f holomorphic on a domain Ω with respect to a sequence $(f \circ \phi_n)$ of compositions, where (ϕ_n) are self-maps of Ω (not necessarily automorphisms).

Universal series. In 1918 Jentzsch gave an example of a power series Σ for which a subsequence of the partial sums of Σ converges outside of its disc D of convergence. Such a power series is said to be overconvergent. Luh, Chui and Parnes showed the existence of such an overconvergent series Σ which is universal in the sense that, for each compact set K in the complement of \overline{D} , and for each f holomorphic on K , there are partial sums of Σ which converge uniformly to f . Nestoridis showed that one can even allow K to meet the boundary of D .

The main focus of the mini-workshop was in fact on universal series of one sort or another. The talks by Wolfgang Luh and Tatevik Gharibyan dealt with the strong relation between universality and lacunarity for power series and also the relation between various forms of summability and holomorphic continuation. Lacunary power series are ones for which many of the coefficients are zero. Another restriction which one can impose on the coefficients is that the sequence of coefficients lie in some sequence space. Vassili Nestoridis, in his first lecture, showed that there are universal series, whose sequence of coefficients are in every ℓ^p -space, for each

$p > 1$. In his second talk, he spoke on the recent work of Mouze concerning universality of the geometric series. Vagia Vlachou gave a talk on universal Faber series. Jürgen Müller, in his talk "From polynomial approximation to universal Taylor series and back again" gave specific examples of the interplay between theorems on approximation by polynomials of a given class and the existence of universal Taylor series whose coefficients satisfy a given restriction.

Potential theory. The extension of some universality results to harmonic functions is due to Armitage (2002, 2003, 2005). Innocent Tamptse's talk was concerned with universal series of harmonic functions. Universality for harmonic functions is based on approximation theory just as universality for holomorphic functions. One reason that harmonic universality was developed much more recently than holomorphic universality is that harmonic approximation theory attained its full development only recently, largely due to Stephen Gardiner. In his talk, Gardiner discussed the relation between approximation theorems and maximum principles in potential theory. The classical approximation theorem of Runge was extended not only to harmonic functions but also to solutions of elliptic partial differential equations (Lax-Malgrange). This leads to universality results for solutions of such equations. Paul Gauthier spoke on universality for solutions of the heat equation (which of course is not elliptic) and also for solutions to Burgers' equation, which is one of the simplest non-linear parabolic equations. Burgers' equation has applications in aerodynamics.

General theory of universality. Several of the talks were not so much concerned with one particular type of universality as with phenomena related to universality in general. For example, the first talk of Nestoridis as well as the talks by Tamptse and Gauthier made use of the recently developed "abstract theory of universality".

A bounded operator T defined on some separable Banach space X is called *hypercyclic* if there exists some vector $x \in X$ such that the orbit of x under T , namely $\{T^n x; n \geq 0\}$ is dense in X . A theorem of Kitai, Gethner and Shapiro asserts that an operator satisfying a certain criterion called the *Hypercyclicity Criterion* is always hypercyclic. The Hypercyclicity Criterion is a very powerful tool to prove that an operator is hypercyclic. And even, until recently, every hypercyclic operator was hypercyclic... because it satisfies the assumptions of the Hypercyclicity Criterion. Thus, a natural question was to know whether every hypercyclic operator satisfies the assumptions of the Hypercyclicity Criterion. Recently, De La Rosa and Read proved that there exist a Banach space X and a hypercyclic operator T on X that does not satisfy the hypercyclicity criterion, but this space cannot be identified with some "classical" Banach space. Frédéric Bayart, in his talk proved that in fact such an operator exists on the separable Hilbert space.

A bounded linear operator T defined on a separable Banach space X is said to be *supercyclic* if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X . It is called *weakly supercyclic* if the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is weakly dense in X .

Fernando León-Saavedra, in his talk proposed a method to prove non-supercyclicity and non-weak supercyclicity which is less computational than previous methods and for which the proofs turn out to be simpler.

George Costakis, spoke on the question as to whether there exist maps between spaces of holomorphic functions which preserve certain notions of universality.

Karl-G. Grosse-Erdmann named his talk "Construction versus Baire category in universality". He could also have named it "Bare hands versus Baire category in universality". The existence of universal functions is usually proved by one of two methods: by an explicit construction or by the use of the Baire category theorem. In his talk, Grosse-Erdmann argued that the two methods are largely equivalent and ended his talk by recalling a pronouncement of T. W. Körner

The Baire Category Theorem is a profound triviality.

The Riemann zeta function. Although universality is a generic phenomenon, the only explicit function which is known to have universality properties is the Riemann zeta function and its close cousins (Voronin, 1975). Markus Nieß extended recent results showing that it is possible to approximate the Riemann zeta function by functions which fail to satisfy the conclusion of the Riemann hypothesis. That is, they do have zeros which are not on the critical axis.

An additional lecture was given by Sophie Grivaux from Lille, who was at Oberwolfach as a participant in the RIP program. She talked about the relation between hypercyclicity and the invariant subset problem.

Problem session. A problem session was held in which problems were presented by Grosse-Erdmann, Aron, Nestoridis, Vlachou, Gauthier, Luh, Gardiner, and Mortini. Gardiner presented some problems in the name of David H. Armitage, who unfortunately could not attend.

Participants found the mini-workshop extremely stimulating. Mathematical and social bonds were reinforced which will surely prolong existing collaborations and develop new ones.

The organizers were Paul M. Gauthier (Montréal), Karl-Goswin Grosse-Erdmann (Mons), and Raymond Mortini (Metz). The participants greatly appreciated the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.