Abstract. The mini-workshop was devoted to the study of hypercyclic and chaotic operators within the wider framework of linear dynamical systems. Topics discussed included common hypercyclic vectors; hypercyclic and supercyclic subspaces; extensions of hypercyclicity like Cesàro-, Faber-, and disjoint hypercyclicity; hypercyclic $N$-tuples and hypercyclic direct sums; hypercyclic $C_0$-semigroups and hypercyclic polynomials; hypercyclicity in non-metrizable spaces; weak supercyclicity; hypercyclic composition operators; and the influence of the norms $\|T^n\|$ on the dynamical behaviour of $T$. A list of open problems is included in the report.

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Introduction by the Organisers

Chaos has long been thought of as being intrinsically linked to non-linearity. The investigations into hypercyclicity in the last two decades have thoroughly refuted this assumption. Many, even quite natural, linear dynamical systems exhibit chaos; this effect, however, only becomes visible when one studies infinite-dimensional state spaces. Starting from the seemingly innocent definition of a hypercyclic operator, that is, an operator with a dense orbit, the theory has developed into a very active research area, the Theory of Linear Dynamical Systems.

In recent years, several open problems have been solved. For example, Ansari and Bernal have given a positive answer to Rolewicz’s question if every separable infinite-dimensional Banach space supports a hypercyclic operator, and Costakis and Peris have given a positive answer to Herrero’s question if every multi-hypercyclic operator is hypercyclic. On the other hand, Herrero’s problem if for every hypercyclic operator $T$ also $T \oplus T$ is hypercyclic has turned out to be a major
challenge. This Great Open Problem in hypercyclicity has, in fact, motivated a fair number of recent investigations.

The solution of another major problem in hypercyclicity was announced by A. Peris during the workshop. In his talk he showed that if \( \{ T_t : t \geq 0 \} \) is a hypercyclic \( C_0 \)-semigroup then every operator \( T_t, t > 0 \), in the semigroup must itself be hypercyclic. This problem had previously evaded the efforts of several researchers.

In an attempt to better understand the Great Open Problem, H. Petersson studied the set of hypercyclic vectors for direct sums of operators. He found conditions under which there are dense subspaces \( U_i, 1 \leq i \leq N \), such that every vector \( (u_i)_{1 \leq i \leq N} \in U_1 \times \ldots \times U_N \) with \( u_i \neq 0 \) for all \( i \) is hypercyclic for \( T_1 \oplus \ldots \oplus T_N \).

Hypercyclic vectors for \( T_1 \oplus \ldots \oplus T_N \) were also studied by J. Bès. However, motivated by an investigation of Furstenberg, he only considered vectors of the form \( (x, \ldots, x) \), and called these vectors d-hypercyclic. He obtained examples of such vectors based on a d-Hypercyclicity Criterion.

In a different direction, N. Feldman called an \( N \)-tuple \( (T_1, \ldots, T_N) \) of commuting operators hypercyclic if the orbit \( \{ T_1^{k_1} \cdots T_N^{k_N} x : k_i \geq 0 \} \) is dense in \( X \) for some vector \( x \). He discussed many examples and presented extensions of results in hypercyclicity to the new setting.

Spectral results in hypercyclicity often involve the unit circle. The talk of S. Grivaux explained why this is so. Replacing the polynomials \( z^n \) by the Faber-polynomials \( F_{\Omega}^n \) of a non-empty simply connected domain \( \Omega \) in \( C \) with compact closure and rectifiable boundary, she defined a vector \( x \) to be \( \Omega \)-hypercyclic for an operator \( T \) if \( \{ F_{\Omega}^n(T)x : n \geq 0 \} \) is dense in \( X \). She presented analogs of results in hypercyclicity in this context, where now the role of the unit disk is taken over by \( \Omega \). In a similar vein, a vector \( x \) is called Cesàro-hypercyclic if the set \( \{ \frac{1}{n} \sum_{k=0}^{n} T^k x : n \geq 0 \} \) is dense in \( X \). For this setting, G. Costakis obtained a version of the somewhere dense orbit theorem of Bourdon and Feldman.

Another topic of the workshop were common hypercyclic vectors. F. Bayart obtained a new condition on a parametrized family \( (T_{\lambda})_{\lambda \in \Lambda} \) to have a dense \( G_{\delta} \)-set of common hypercyclic vectors. The proof used both the Baire category theorem and a nontrivial result from probability theory. A different sufficient condition for the same problem was given by K. Chan. He employed his condition to show that any two hypercyclic unilateral weighted backward shifts can be connected by a path of such operators having a dense \( G_{\delta} \)-set of common hypercyclic vectors.

H. N. Salas studied the existence of supercyclic subspaces, that is, closed infinite-dimensional subspaces of supercyclic vectors. He presented two sufficient conditions for their existence, and studied their necessity in the special case of backward shift operators. K.-G. Grosse-Erdmann extended the known sufficient condition for the existence of hypercyclic subspaces to the setting of \( F \)-spaces, and he obtained an analogous result on frequently hypercyclic subspaces.

The topic of supercyclicity was also taken up by É. Matheron. He first showed that unitary operators can be weakly supercyclic; his example was the multiplication operator \( M_\mu \) on a space \( L^2(\mu) \), where \( \mu \) is a probability measure on \( T \) of small
support. He then presented recent work of S. Shkarin saying that the measure $\mu$ may even be such that its Fourier coefficients tend to 0.

J. Bonet studied hypercyclicity and chaos of the differentiation operator on some natural spaces of holomorphic functions. The special interest of these results was that he considered spaces that are non-metrizable, so that the Baire category theorem is not available. He also presented a characterization of topological transitivity and chaos of the backward shift on non-metrizable sequence spaces.

F. Martínez characterized hypercyclicity and chaos of certain polynomials on Köthe echelon spaces. He also discussed a result that links the infinite-dimensional dynamics of polynomials to the Julia set of a corresponding complex polynomial.

Two of the talks studied the behaviour of the sequence of norms $\|T^n\|, n \geq 1$, of operators $T$ on Banach spaces. C. Badea characterized those sequences $(u_n)$ such that, for any operator $T$ on any separable Banach space $X$, $\sup_{k \geq 0} \|T^{u_k}\| < \infty$ implies that the unimodular point spectrum $\sigma_p(T) \cap T$ is at most countable. As an application he obtained an example of a chaotic and frequently hypercyclic operator on Hilbert space that is not topologically mixing, thereby answering a question of Peris. V. Müller considered an opposite of hypercyclicity, vectors $x$ for which $\|T^n x\|$ is big in some sense. He showed that, for any sequence $(T_n)$ of operators and any positive sequence $(a_n)$ with $\sum_n a_n < \infty$ there is a vector $x$ so that $\|T^n x\| \geq a_n \|T_n\|$ for all $n \geq 0$. He also obtained analogues for weak orbits.

R. Mortini investigated composition operators $f \mapsto f \circ \phi_n, n \geq 1$, on the space $B = \{ f \in H(\Omega) : \sup_{z \in \Omega} |f(z)| \leq 1 \}$, where $\Omega$ is a domain in $\mathbb{C}^N$ and the $\phi_n$ are automorphisms or, more generally, holomorphic self-maps of $\Omega$. He characterized when there exists an $f \in B$ so that $\{f \circ \phi_n : n \geq 1\}$ is locally uniformly dense in $B$, and also presented an analogous result when $B$ is replaced by the space $H(\Omega)$ of all holomorphic functions on $\Omega$.

It is a distinctive feature of hypercyclicity that it unites researchers from various backgrounds, be it topological dynamics, operator theory, semigroup theory, the theory of locally convex spaces or complex analysis, which was also reflected in the composition of the participants. A very lively exchange of ideas characterized the mini-workshop, which was particularly apparent in the two Special Sessions that both ran overtime. Some of the problems discussed in the Problem Session on Wednesday morning and some of the contributions presented in the Informal Session on Friday morning are collected at the end of this report, while additional problems can be found in the abstracts of J. Bés, N. Feldman, F. Martínez, H. Petersson and H. N. Salas.

The mini-workshop was organized by Teresa Bermúdez (La Laguna), Gilles Godefroy (Paris), Karl-G. Grosse-Erdmann (Hagen), and Alfredo Peris (Valencia). Unfortunately, Teresa Bermúdez was unable to participate. The participants greatly appreciated the hospitality and the stimulating atmosphere of the Forschungsinstitut Oberwolfach.