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Mathematical General Relativity

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Abstract. General Relativity is one of the triumphs of twentieth century physics. Its spectacular predictions include gravitational waves, black holes, and spacetime singularities. The mathematical study of this theory leads to deep problems connecting the areas of partial differential equations, geometry and topology with physics. The talks of the workshop illustrated the rapid progress in this subject over the last few years.

Mathematics Subject Classification (2010): 83-XX, 35-XX, 53-XX.

Introduction by the Organisers

The workshop Mathematical Aspects of General Relativity was organised by Carla Cederbaum (Tübingen), Mihalis Dafermos (Cambridge/Princeton), Jim Isenberg (Eugene) and Hans Ringström (Stockholm). The over 50 participants represented a wide selection of different research areas connected to the general theory of relativity, and roughly half of them gave talks at the workshop.

Proving nonlinear stability for solutions of the vacuum Einstein equations (without symmetry assumptions) has been a long-term goal of the field, and the progress reported in previous workshops is now finally converging to a complete solution. In this direction, Martin Taylor discussed a proof (in progress, joint with Holzegel, Rodnianski and Dafermos) of the full nonlinear stability of the Schwarzschild solution, the first non-trivial asymptotically flat black hole solution to be discovered. Andras Vasy described his proof (joint with Hintz) of stability of very slowly rotating Kerr-de Sitter spacetimes (for the case of positive cosmological constant). Peter Hintz discussed a new proof of the non-linear stability of Minkowski space.

Finally, on this theme, Pau Figueras presented beautiful numerical work exhibiting various novel *instabilities* of higher dimensional black holes. These numerics may serve as inspiration for mathematical work to be presented at future workshops!

The Einstein–Vlasov and Vlasov–Poisson systems are of central interest in physics, since they can be used to model, e.g., galaxies. One central question concerning solutions to these systems is whether they are stable or not. This question was addressed in a talk by Mahir Hadzic. In particular, he described an instability theory for self gravitating relativistic matter systems.

In addition to stability problems, important progress has been made in understanding solutions of the vacuum Einstein equations with no (or limited) symmetry in various singular regimes. Jonathan Luk presented joint work with Van de Moortel concerning the interaction of impulsive gravitational waves with *three* singular fronts, while Y. Shlapentokh-Rothman presented joint work with Rodnianski concerning asymptotically self-similar solutions.

As discussed in several previous workshop of this series, Anti-de Sitter space has been conjectured to be *unstable*, in stark contrast to Minkowski and de Sitter space. Though numerical evidence has more recently been given for the validity of this conjecture, it has remained elusive to prove because the instability mechanism is a purely non-linear effect. Georgios Moschidis discussed the first rigorous proof of this instability, accomplished in the context of the Einstein–Vlasov system under spherical symmetry. On the other hand, Stephen Green discussed numerical work connected to so called "islands of stability".

Black hole interiors remain one of the most intriguing domains for general relativity. Sung-Jin Oh discussed his joint work with Luk concerning *weak null singularities* in black hole interiors, giving finally a complete proof of a suitable formulation of Strong Cosmic Censorship for the Einstein–Maxwell-scalar field eqations under spherical symmetry. Christoph Kehle presented joint work with Shlapentokh-Rothman concerning a scattering theory for the wave equation on Reissner–Nordström.

The formulation of strong cosmic censorship requires the notion of "maximal Cauchy development". In his talk, Jan Sbierski explained the subtleties in defining this object for some of the most classical equations of mathematical physics.

The causal structure plays a central role in the understanding of the asymptotic behaviour of solutions to Einstein's equations. In the standard models of the universe, particle horizons form in the direction of the big bang. However, already in 1969, Misner suggested that in, e.g., Bianchi type IX solutions, particle horizons would not form. In recent work, presented during the workshop, Bernhard Brehm demonstrated that for Lebesgue almost all Bianchi type IX initial data, particle horizons form. However, he also presented heuristics indicating that for Baire generic initial data, particle horizons do not form.

Back reaction is a topic of current interest in physics. During the workshop it was discussed in the presentation by Cécile Huneau. In joint work with Jonathan Luk, she has explored the relation between weak limits of solutions to Einstein's vacuum equations and solutions to the Einstein null dust system. In particular,

Huneau and Luk have demonstrated that (under suitable assumptions) solutions to the Einstein null dust system can be obtained as weak limits of families of solutions to Einstein's vacuum equations. One of the basic examples of this phenomenon is obtained in the class of polarised Gowdy solutions. A generalisation of this class is the T^2-symmetric solutions. The future asymptotics of solutions to this family was discussed in the presentation of Adam Layne. In joint work with Beverly Berger and James Isenberg, he has identified an attractor for the future asymptotics. However, the attractor does not represent a known explicit solution.

Gregory Galloway presented a result joint with Eric Ling giving an interesting connection between the topology of cosmological spacetimes and the occurrence of singularities. Roughly speaking, the result states that in a globally hyperbolic spacetime satisfying the null energy condition and having a smooth compact Cauchy hypersurface with a positive definite second fundamental form, the following holds. Either the Cauchy hypersurfaces are spherical spaces or the spacetime is past causally geodesically incomplete. In a related talk, Melanie Graf presented singularity theorems for $C^{1,1}$-metrics. A topological understanding of photon spheres in Kerr spacetimes was presented by Sophia Jahns, relying on a lift to the phase space and using methods related to some of those of central importance in the result presented by Galloway (joint with Carla Cederbaum).

Henri Roesch discussed his result on the Penrose inequality for (rather) general null hypersurfaces in 4-dimensional spacetimes. His method of proof heavily relies on a new quasi-local notion of mass that is tailored to converge to Bondi mass and has very useful monotonicity properties.

A number of talks focused on initial data for Einstein's equations with prescribed asymptotic behavior. Many talks focused on the special case of Riemannian manifolds, typically assuming that those have non-negative scalar curvature, corresponding to the dominant energy condition in the spacetime generated from the Riemannian manifold, assuming time symmetry.

In this context, Richard Schoen presented his groundbreaking result with Shing-Tung Yau settling the positive mass theorem for asymptotically flat Riemannian manifolds obeying the dominant energy condition (non-negative scalar curvature, in this setup) in arbitrary dimension without any topological conditions. Working instead in the asymptotically hyperbolic context, but still in arbitrary dimension, Romain Gicquaud showed that certain algebraic ideas can be used to define novel, mass-like covariants to be determined from the asymptotics of the metric (joint with Julien Cortier and Mattias Dahl).

Christina Sormani (joint work with a number of co-authors) and Jeffrey Jauregui (joint work with Dan Lee) each presented subtle results on stability / almost rigidity of general relativistic inequalities for Riemannian manifolds such as the positive mass theorem, using the relatively new, tailor-made notion of intrinsic flat convergence suggested by Sormani and Wenger.

Working with methods from Riemannian geometry and also with those developed in the context of curvature flows, Martín Reiris presented an extensive classification result for static spacetimes with an axisymmetry condition but without any asymptotic assumptions.

Furthermore, Armando Cabrera Pacheco presented a gluing construction for Riemannian extensions allowing to give an upper estimate on the quasi-local Bartnik mass of a 2-dimensional surface with prescribed metric and mean curvature, defined as the infimum of the ADM-masses of all "admissible" Riemannian manifolds realizing the given geometric data on the 2-surface. This construction extends and refines a construction suggested by Christos Mantoulidis and Richard Schoen for minimal surfaces in the asymptotically flat case both to the context of constant mean curvature surfaces and to the asymptotically hyperbolic case (joint with Carla Cederbaum, Stephen McCormick, Pengzi Miao).

More generally, for arbitrary asymptotically flat initial data sets satisfying the dominant energy condition, several results were presented that demonstrate the progress the field has made in the last years: Both Ye Sle Cha (partially joint work with Marcus Khuri) and Eugenia Gabach Clement (partially joint work with Sergio Dain and also with others) discussed recent progress made towards proving geometro-physical inequalities such as the quasi-local area-angular momentum and global angular momentum-mass inequalities for black holes.

Anna Sakovich presented a novel foliation near infinity for asymptotically flat initial data sets in three dimensions that generalizes the canonical foliation by constant mean curvature surfaces initially studied by Gerhard Huisken and Shing-Tung Yau and that is intimately linked with the mathematically consistent definition of the center of mass of an isolated system (joint with Carla Cederbaum).

The participants of the workshop benefitted a lot from an open problem session led by Robert Wald, where he discussed the subtle intricacies of well-posing perturbative corrections to surprisingly simple equations.

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Workshop: Mathematical General Relativity

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