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Calculus of Variations

Organised by
Alessio Figalli, Zürich
Robert V. Kohn, New York
Tatiana Toro, Seattle
Neshan Wickramasekera, Cambridge UK

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Abstract. The Calculus of Variations is at once a classical subject, and a very modern one. Its scope encompasses a broad range of topics in geometric analysis, and deep questions about PDE. New frontiers are constantly emerging, where problems from mechanics, physics, and other applications introduce new challenges. The 2018 Calculus of Variations workshop reflected this breadth and diversity.

Mathematics Subject Classification (2010): 49-xx [Calculus of Variations] 35Jxx [Elliptic equations and systems] 53Cxx [Global differential geometry] 58xx [Global analysis, analysis on manifolds].

Introduction by the Organisers

This Calculus of Variations workshop was organized by Alessio Figalli (Zurich), Robert Kohn (New York), Tatiana Toro (Seattle), and Neshan Wickramasekera (Cambridge). It gathered an outstanding group of 47 participants, among them many PhD students and postdocs.

The workshop's scope was very broad. The diversity of the topics represented encouraged scientific cross-fertilization and was a key to the workshop's success. To capture the diversity – and coherence – of the workshop's 24 talks, we shall organize them into groups.

Three talks discussed **free boundary problems**. Two speakers addressed thin obstacle problems: *Emanuele Spadaro* presented work with M. Focardi about graphs that minimize surface area subject to the constraint of being above a flat, lower-dimensional obstacle; the work establishes sharp regularity results for the solutions by reducing the question to previously known results on branched minimal graphs. *Angkana Rüland* presented work with H. Koch and W. Shi concerning variable-coefficient scalar analogues of the classic Signorini problem. A third talk, by *Charles Smart*, presented work with Will Feldman on an evolutionary free boundary problem modeling the motion of the contact line where a spreading liquid drop meets a periodically patterned surface. Starting from a discrete Hele-Shaw-type formulation, Smart discussed its scaling limit and explained why the solution has facets (a phenomenon also seen experimentally).

Quite a few talks discussed problems from geometric analysis. One thread involved the **asymptotic character of solutions near singularities**. *Nick Edelen* discussed work with M. Colombo and L. Spolaor establishing an asymptotic decay estimate for stationary varifolds close to an integrable multiplicity 1 polyhedral cone. *Max Engelstein* described work with L. Spolaor and B. Velichkov establishing log-epiperimetric inequalities and asymptotic decay results for area minimizers and energy minimizing free boundaries at strongly isolated singularities. A key step of the proof involves the use of the classical (finite dimensional) Lojasiewicz inequality. *Brian Krummel* described work with N. Wickramasekera establishing asymptotic decay of a Dirichlet energy minimizing multivalued function to a unique tangent function a.e. along its branch locus. A key idea in this work is the use of the Almgren frequency function to classify homogeneous "Jacobi fields" produced by sequences of minimizers converging to a homogeneous minimzer, even though these Jacobi fields themselves need not be energy minimizing or even stationary in the usual sense.

Another thread from geometric analysis was the use of **min-max techniques** for constructing and analyzing minimal surfaces. *Christos Mantoulidis* described work with O. Chodosh giving a PDE-based proof of regularity of two-dimensional Allen-Cahn min-max minimal surfaces in 3-manifolds; for generic metrics the resulting solutions surfaces have multiplicity 1. *Alessandro Pigati* described work with T. Rivière showing regularity of the "parameterised stationary 2-varifolds" in arbitrary co-dimension; these arise from a novel min-max construction (due to Rivière) for a certain perturbation of the mapping area.

A third thread from geometric analysis was the study of **geometric flows**. *Brian White* presented work with D. Hoffman, T. Ilmanen, and F. Martín concerning "translator" solutions of the mean curvature flow (in other words: hypersurfaces M such that the translating surface $M - te_{n+1}$ is a solution of the mean curvature). *Jacob Bernstein* described recent progress with L. Wang concerning asymptotically conical self-expanders for the mean curvature flow; the work adapts global analysis techniques used previously in minimal surface theory to self-expanders, and establishes a certain non-degeneracy property of the expanders asymptotic to a generic regular cone. *Melanie Rupflin* presented work with P. Topping and with C. Robertson describing finite time degeneration of Teichmüller harmonic map flow.

There were also talks on **other aspects of geometric analysis**. One, by *Francesco Maggi*, described work with M. Delgadino, C. Mihailia, and R. Neumayer establishing that unions of spheres are the only finite-volume Caccioppoli sets that are stationary for the isoperimetric problem in Euclidean space. Another, by *Guy David*, discussed various notions of a "solution" to the classical Plateau problem, focusing especially on a definition involving the "sliding boundary condition" and on boundary regularity of the associated class of almost minimal sets. *Antonio De Rosa*'s talk presented work with G. De Philippis and F. Ghiraldin giving an extension of the Allard rectifiability theorem to anisotropic integrands. *Dali Nimer*'s talk discussed uniformly distributed measures, presenting new examples and characterisations of conical 3-uniform measures.

About half the talks were on topics other than geometric analysis. **Lower semicontinuity of functionals** and the **regularity of minimizers** are familiar topics in the calculus of variations, and three speakers discussed problems of this type. *Yury Grabovsky* presented a new example of a variational problem that is rank-one convex but not quasiconvex, obtained by using connections between homogenization, optimal design, and quasiconvexity, combined with an algebraic approach to "exact relations" for polycrystalline composite materials. *Connor Mooney* presented new results on the regularity of solutions to elliptic and parabolic systems; in the parabolic setting, a key idea was to look for a "spiraling self-similar" solution. *Felix Otto* presented a new variational approach to regularity for the Monge-Ampere equation (work with M. Goldman and M. Huesmann); it uses the connection between Monge-Ampere and optimal transport, and well-chosen test functions in the Benamou-Brenier variational formulation of optimal transport.

In many physical applications, variational problems must be solved in settings where the spatial environment is highly heterogeneous; there were four talks of this type, where the calculus of variations interacted with **homogenization**. Two (by Charles Smart and by Yury Grabovsky) were already discussed above. A third, by *Ken Golden*, discussed how homogenization is critical to our understanding of sea ice in the arctic and antarctic, improving our understanding of how global warming will affect the climate and raise the level of the sea. Another talk, by *Caterina Zeppieri*, discussed work with F. Cagnetti, G. Dal Maso, and L. Scardia on a family of stochastic homogenization problems motivated by the modeling of fracture.

Another application-driven frontier is the modeling of **thin elastic sheets**; there were three talks in this area. *Heiner Olbermann* discussed this topic's deep connections to the Nash embedding theorem, then presented recent results explaining why a sheet with a conical singularity is rather rigid. *Marta Lewicka* discussed work with D. Lucic, using methods from Γ-convergence to see how prestrain (due, for example, to growth or thermal expansion) affects the mechanical behavior of a thin elastic sheet. *Ian Tobasco* discussed the wrinkling seen when a piece of a thin spherical shell is forced to be (approximately) flat by putting it on water, identifying a regime where the energy of the sheet (suitably renormalized)

Γ-converges to a convex variational problem, and drawing conclusions about the associated wrinkling patterns.

Yet another application-driven frontier is **energy-driven pattern formation**, where complex patterns emerge from the solutions of variational problems. Tobasco's talk (just discussed) had this character. So did the one by *Benedikt Wirth*, who discussed the numerical approximation of "branched transport" problems – identifying, for a broad class of such problems, a family of diffuse-interface approximations that are well-suited to numerical minimization.

The diversity of the workshop's topics was an important element in its success. Experts in one area enjoyed looking for and finding connections to the others. A comprehensive list of examples is beyond the scope of this Introduction, but here are a few examples: (i) some geometric analysts wondered whether Yury Grabovsky's rank-one-convex but non-quasiconvex integrand might have a geometric interpretation; (ii) experts in the mean curvature flow were interested to see self-similarity (or something very close to it) playing a role not only in the analysis of topological change but also in Connor Mooney's results on singular solutions of other parabolic systems; and (iii) people working on steady-state homogenization problems took great interest in the homogenization-like evolutionary free boundary problem discussed by Charles Smart.

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