Copositivity and Complete Positivity

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Abstract. A real matrix $A$ is called copositive if $x^T Ax \geq 0$ holds for all $x \in \mathbb{R}_+^n$. A matrix $A$ is called completely positive if it can be factorized as $A = BB^T$, where $B$ is an entrywise nonnegative matrix. The concept of copositivity can be traced back to Theodore Motzkin in 1952, and that of complete positivity to Marshal Hall Jr. in 1958. The two classes are related, and both have received considerable attention in the linear algebra community and in the last two decades also in the mathematical optimization community. These matrix classes have important applications in various fields, in which they arise naturally, including mathematical modeling, optimization, dynamical systems and statistics. More applications constantly arise.

The workshop brought together people working in various disciplines related to copositivity and complete positivity, in order to discuss these concepts from different viewpoints and to join forces to better understand these difficult but fascinating classes of matrices.


Introduction by the Organisers

A real matrix $A$ is called copositive if $x^T Ax \geq 0$ holds for all $x \in \mathbb{R}_+^n$. Obviously, every positive semidefinite matrix is copositive, and so is every entrywise symmetric nonnegative matrix. However, for $n \geq 5$ the cone of $n \times n$ copositive matrices is considerably larger and more complex than both the semidefinite and symmetric nonnegative matrix cones. Its dual cone is the cone of $n \times n$ completely positive matrices, that is, matrices that have a representation $A = \sum b_i b_i^T$ with $b_i \in \mathbb{R}_+^n$. 
The cone of completely positive matrices is contained in the cone of doubly non-negative matrices, i.e., matrices which are both positive semidefinite and entrywise nonnegative. For \( n \leq 4 \) these two cones are equal \[35\], but for \( n \geq 5 \), there are \( n \times n \) doubly nonnegative matrices which are not completely positive. Both the copositive and the completely positive matrix cones are closed, convex, full dimensional and pointed \[4\]. They are, however, not polyhedral, but rather their boundaries have both polyhedral parts and “curved parts.” For the copositive cone, a characterization in terms of its supporting hyperplanes is known, but a complete characterization of its extremal (generating) rays is an open question. For the completely positive cone, the converse is true: the extremal rays are known, but characterizing the supporting hyperplanes is an open problem. Checking membership in each of these cones is (theoretically and algorithmically/computationally) challenging. These open problems and several others are highlighted in \[3\].

Both matrix cones possess highly interesting properties, and have attracted interest in the linear algebra community for many years. For surveys on copositive matrices see \[31, 30\] and for one on completely positive matrices see \[7\].

Several necessary and sufficient conditions for copositivity are known. Schur complement properties have been proposed \[11\], special subclasses of copositive matrices have been characterized (e.g., matrices whose entries are \( \pm 1 \) or 0, cf. \[29\]), spectral properties have been studied, and a partial characterization of extremal copositive matrices has been given. The extremal \( 5 \times 5 \) copositive matrices have recently been fully described \[28\]. A nonnegative vector \( x \) is a zero of a copositive matrix \( A \) if \( x^T Ax = 0 \). These vectors play an important role in the study of extreme copositive matrices \[27\]. It is known that testing copositivity is a co-NP-complete problem \[37\]. Nonetheless, several algorithmic copositivity tests have been developed \[16, 13, 46, 44, 19\].

On the dual side, necessary and sufficient conditions for complete positivity have also been introduced. Often, these involve the zero-nonzero pattern of the matrix, described by a graph. For example, the necessary condition of being doubly nonnegative was shown to be also sufficient if and only if the graph of the matrix has no long odd cycle \[5, 4, 33, 1\]. A sufficient condition in terms of the so-called comparison matrix of the given matrix was introduced, and shown to be also necessary when the graph of the matrix is triangle free \[23\]. Finding a representation \( A = \sum_{i=1}^{r} b_i b_i^T \) with \( b_i \in \mathbb{R}^n_+ \) is an open question even in cases where the matrix \( A \) is known to be completely positive. The minimal number \( r \) of rank-one summands needed in such a representation is called the cp-rank of \( A \), see \[7\].

Upper bounds for the cp-rank of matrices are known, and it was conjectured \[23\] that \( \text{cp-rank}(A) \leq \frac{n^2}{4} \) for any \( n \times n \) completely positive matrix \( A \). This bound has been shown to hold in some special cases \[23, 8, 44, 42, 20\]. However the conjecture in general has been recently refuted, and an asymptotic bound has been given \[15, 14\]. Testing a rational matrix for complete positivity was shown to be NP-hard \[22\]. Although it is an open question to determine a representation \( A = \sum_{i=1}^{r} b_i b_i^T \) with \( b_i \in \mathbb{R}^n_+ \) of a general completely positive matrix \( A \), some algorithms have been suggested \[21, 39, 2, 26, 45, 10\]. The computational question...
of determining whether or not a given symmetric nonnegative matrix is completely positive is studied in [9].

From the above, it is clear that copositivity and complete positivity are highly interesting concepts from the point of view of linear algebra. They also have many important applications. One of the early motivations to study complete positivity was its relevance to block designs. Other applications include a maximin efficiency-robust test, a proposed mathematical model of energy demand, exchangeable probability distributions on finite sample spaces and a Markovian model of DNA evolution [7]. Some recent applications include hard and probabilistic clustering [47], scheduling of stochastic arrivals [52], and dynamical systems [6]. One of the most important applications is to mathematical optimization, which is briefly described below.

Many combinatorial and nonconvex quadratic optimization problems have been shown to possess an equivalent formulation as linear problems over the copositive or completely positive cone. This was first shown for the so-called standard quadratic programming problem. Letting $Q \in \mathbb{R}^{n \times n}$ be an arbitrary symmetric matrix and denoting by $e$ the all-ones vector, it has been shown by Bomze et al. [12] that any nonconvex quadratic problem over the simplex

$$\min \{ x^T Q x : e^T x = 1, x_i \geq 0 \text{ for all } i \},$$

has an equivalent completely positive formulation (with $E = ee^T$)

$$\min \{ \langle Q, X \rangle : \langle E, X \rangle = 1, X \text{ is completely positive} \}.$$

This equivalence is remarkable, since it transforms a nonconvex NP-hard optimization problem into a linear problem in matrix variables over a convex cone of matrices. The local minima of (1) disappear, and there is a one-to-one correspondence between the global minima of (1) and the local (= global) minima of (2). The difficulty of the problem is thus shifted entirely into the cone constraint, which makes understanding the cone crucial for tackling the problem.

Generalizing this result, Burer [17] showed that any quadratic problem with linear and binary constraints

$$\min \ x^T Q x + 2e^T x$$

$$\text{st.} \quad a_i^T x = b_i \quad (i = 1, \ldots, m)$$

$$x \geq 0, \quad x_j \in \{0, 1\} \quad (j \in B),$$

with $Q$ not necessarily positive semidefinite and $B \subset \{1, \ldots, n\}$, can equivalently be written in the form

$$\min \ \langle Q, X \rangle + 2e^T x$$

$$\text{st.} \quad a_i^T x = b_i \quad (i = 1, \ldots, m)$$

$$\langle a_i a_i^T, X \rangle = b_i^2 \quad (i = 1, \ldots, m)$$

$$x_j = X_{jj} \quad (j \in B)$$

$$\begin{pmatrix} 1 & x \\ x & X \end{pmatrix} \text{ is completely positive},$$

which is again a linear problem over the cone of completely positive matrices. Because of the binary constraints, this latter setting includes a broad class of
Combinatorial problems and shows that most of them can be formulated as linear problems over the completely positive cone.

Combinatorial problems for which copositive formulations have been studied include the clique number [12], the chromatic number [25], the quadratic assignment problem [40], and certain graph partitioning problems [41].

Note that the dual problem of a completely positive problem is a problem over the dual cone, i.e., the copositive cone, and vice versa. Under mild regularity conditions, strong duality holds between the problems, thus any progress in understanding either of the cones will help solving these difficult optimization problems.

We mention that copositivity also plays a role in modeling optimization under uncertainty [38], in complementarity problems [18], and matrix games [36]. For more details, we refer to the survey papers [24] and [30].

As outlined above, copositivity and complete positivity are highly interesting and relevant topics that have attracted increasing attention in the last two decades. These topics are of interest in various specialty areas, both in pure and applied mathematics. The workshop brought together researchers working in different areas where copositivity and complete positivity arise, including some who had not met in person before. The following abstracts give an idea about the different views onto this key concept and the fruitful discussions that followed.

References


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