Abstract. There are several flavors of positivity in Algebraic Geometry. They range from conditions that determine vanishing of cohomology, to intersection theoretic properties, and to convex geometry. They offer excellent invariants that have been shown to govern the classification and the parameterization programs in Algebraic Geometry, and are finer than the classical topological ones. This mini-workshop aims to facilitate research collaboration in the area, strengthening the relationship between various positivity notions, beyond the now classical case of divisors/line bundles.

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Introduction by the Organisers

The mini-workshop Positivity in Higher-dimensional Geometry brought together algebraic geometers with various interests circling around the idea of positivity. The participants invested the majority of their time in group work on open questions related to positivity of higher-codimensional numerical cycle classes on projective varieties, or convex geometry in the form of Newton–Okounkov bodies. There were also seven research talks given by Jian Xiao (Evanston), Victor Lozovanu (Hannover), Chung Ching Lau (Chicago), John Christian Ottem (Oslo), Catriona Maclean (Grenoble), Stefano Urbinati (Padova), and Nguyen-Bac Dang (Paris) in this order. The extended abstracts of their presentations appear in the sequel.

The topic of positivity is an active research subject in Algebraic Geometry:
The position of the canonical class $K_X$ of a projective variety $X$ relative to the ample cone $\text{Amp}(X) \subset N^1(X)$ guides the Minimal Model Program.

In Moduli Space theory, stability is behind boundedness for parameter spaces for varieties or for sheaves on them with prescribed numerical invariants.

Most of the techniques in contemporary research involve classical positivity properties of divisors/line bundles, or, by duality, curve classes. The primordial notion is ampleness. It can be characterized cohomologically, intersection theoretically (cf. [Kle66]), and geometrically. A geometric generalization is bigness. A divisor $D$ is big if it is the sum $D = A + E$ of an ample divisor $A$ and an effective divisor $E$. Its natural cohomological characterization involves an asymptotical construction:

$$D \text{ is big } \iff \text{vol}(D) > 0,$$

where $\text{vol}(D)$ measures the asymptotic growth of the dimension of the linear series $|mD|$ as $m$ grows. More precisely

$$\text{vol}(D) \equiv (\dim X)! \cdot \limsup_{m \to \infty} \frac{\dim H^0(X; \mathcal{O}_X(mD))}{m^{\dim X}}.$$

Recently, a convex geometric approach (cf. [LM09, KK12]) through the theory of Newton–Okounkov bodies has provided the ideal package for much of this information. To a complete flag of subvarieties

$$Y_* : X = Y_0 \supset Y_1 \supset \ldots \supset Y_{\dim X} = \{x\} \in X$$

centered a closed point $x$ that is smooth for all $Y_i$, one associates a rank $\dim X$ valuation-type function on the global sections on $\mathcal{O}_X(mD)$ for all $m$ and a convex body $\Delta_{Y_*}(D) \subset \mathbb{R}^{\dim X}$. It encodes many of the positivity properties of $D$:  

- $\text{vol}(D)$ is the normalized Euclidean volume of the Newton–Okounkov body $\Delta_{Y_*}(D) \subset \mathbb{R}^{\dim X}$, cf. [LM09],
- The divisorial Zariski decomposition of $D$ can be computed (cf. [Jow10, KL17, CHPW15]), from the knowledge of $\Delta_{Y_*}(D) \subset \mathbb{R}^{\dim X}$ for all flags $Y_*$ on $X$ as above.
- The restricted volumes $\text{vol}_{X|V}(D)$ can be determined (cf. [LM09]) from the knowledge of $\Delta_{Y_*}(D) \subset \mathbb{R}^{\dim X}$ for all flags $Y_*$ on $X$ as above.
- The Seshadri constant $\varepsilon(D; x)$ can be found (cf. [KL17]) by working with infinitesimal flags centered at $x$.

In higher (co)dimension, historically the outlook has been negative and significant progress in the way of general results (as opposed to pathological examples) is of a more recent nature. There are two directions here: [Ott12, Lau16] study positivity from a cohomological perspective. A smooth subvariety $V \subset X$ of dimension $d$ of a complex projective manifold is said to be ample if the relative $\mathcal{O}(1)$ on $\text{Bl}_V X$ satisfies certain cohomology vanishing conditions (cf. [Ott12]). For example [Ott12] recovers Lefschetz-hyperplane-type results, while [Lau16] proves versions of Fujita vanishing in this context.

A numerical intersection theoretic perspective is adopted by Fulger–Lehmann in [FL17a, FL17b]. The mantra is that the geometry of cycles should be reflected
in the geometry of convex cones inside the numerical groups $N_d(X)$. Classical examples of such cones are the pseudo-effective cone $\overline{\text{Eff}}_d(X) \subset N_d(X)$, the closure of the convex cone generated by $d$-dimensional subvarieties of $X$. Another example is its dual $\text{Nef}^d(X) \subset N^d(X)$ of classes $\beta \in N^d(X)$ with $\beta(\alpha) \geq 0$ for all $\alpha \in \overline{\text{Eff}}_d(X)$.

When $d = 1$, by [Kle66] the ample cone $\text{Amp}(X)$ is the interior of the nef cone of divisors $\text{Nef}^1(X) \subset N^1(X)$. In particular $\text{Nef}^1(X) \subset \overline{\text{Eff}}_{\text{dim} X - 1}(X)$. This inclusion may fail for arbitrary $d$ (cf. [DELV11]). Thus we may not expect good geometry from an arbitrary nef class. However [FL17a] prove that $\text{Nef}^d(X)$ is full-dimensional in $N^d(X)$ and contains complete intersection classes of ample divisors in its strict interior. As a corollary they show that $\overline{\text{Eff}}_d(X) \subset N_d(X)$ is a pointed cone.

The main goal of the workshop was to form diverse groups focused on open questions relating to cycles and/or convex geometry, facilitating future collaboration on the subject. The participants split into four groups, and their assignments are described below.

**Problem 1.**

1. Is the convex cone generated by classes of ample subvarieties open in $N_d(X)$?
   This is an attempt to understand the relation between cohomological and numerical positivity in higher codimension, paralleling known results for divisor classes.

2. Is there a convex geometric approach to higher cohomology functions similar to the Newton–Okounkov interpretation of $\text{vol}(D)$? If $D$ is a divisor on a projective variety, is there an object in convex geometry whose Euclidean volume is naturally equal to
   
   \[ \hat{h}^i(D) \overset{\text{def}}{=} (\text{dim } X)! \cdot \limsup_{m \to \infty} \frac{\dim H^i(X; \mathcal{O}_X(mD))}{m^{\text{dim } X}}, \]

   Can lim sup be replaced by lim? If this is true, then the following also holds
   
   \[ (D^{\text{dim } X}) = \sum_{i=0}^{\text{dim } X} (-1)^i \hat{h}^i(D). \]

3. Let $X$ be a smooth toric variety and let $T \subset X$ be the open torus. The movable $d$-dimensional subvarieties of $X$ are those that admit deformations through points of $T$. Is the closed cone $\text{Mov}_d(X) \subset N_d(X)$ generated by numerical classes of such subvarieties rational polyhedral?
   The cases of divisors and curves are known to be true, but their proofs rely on deep results that have no clear generalizations to arbitrary (co)dimension. The movable cone of divisors is rational polyhedreal because toric varieties are Mori Dream Spaces (cf. [HK10]). The movable cone of curves is dual to the effective cone of divisors (cf. [BDPP13] in the general case. A more elementary proof in the toric case appears in [Pay06]), which is easily seen to
be rational polyhedral.

(4) Work of Lehmann [Leh16] and Lehmann–Xiao [LX16] has produced a good theory of positivity for curve classes in $N_1(X)$ by duality to the case of divisors. Can this be extended to a Newton–Okounkov-type convex body picture?

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Higher-codimensional Cycles and Newton–Okounkov Bodies

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