Abstract. The aim of this Arbeitsgemeinschaft is to go over the proof of the higher Gross–Zagier formula established in the paper [YZ15]. The formula relates arbitrary order central derivative of the base change $L$-function of an unramified automorphic representation of $\text{PGL}_2$ over a function field to the self-intersection number of a certain algebraic cycle on the moduli stack of Shtukas.

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Introduction by the Organisers

For an elliptic curve $E$ over a global field $K$, the conjecture of Birch and Swinnerton-Dyer asserts a deep relationship between the arithmetic invariants (Mordell–Weil groups and Tate–Shafarevich groups) and the analytic invariants (the complex $L$-function $L(E/K,s)$). The rank part of the conjecture asserts that the vanishing order of $L(E/K,s)$ at its center $s = 1$ coincides with the rank of Mordell–Weil group $E(K)$. The refined part of the conjecture is an identity of the leading term of $L(E/K,s)$ at $s = 1$,

$$L^{(r)}(E/K,1) \frac{r!}{r!} \sim \det ((P_i, P_j)_{\text{NT}})$$

where $(P_i, P_j)_{\text{NT}}$ is the matrix of Néron–Tate height pairings of a $\mathbb{Z}$-basis $\{P_1, ..., P_r\}$ of $E(K)/E(K)_{\text{tor}}$, and $\sim$ means the two sides are equal up to some explicit terms such as the order of Tate–Shafarevich group, the local Tamagawa numbers and the real periods. Equivalently, the Néron–Tate height pairing induces a metric on the determinant of the Mordell–Weil group $E(K) \otimes \mathbb{Z} \mathbb{R}$, and the RHS of the conjectural formula is the norm of a generator of the determinant of
the lattice $E(K)/E(K)_{tor}$. Beilinson and Bloch also formulated a generalization of the B-SD conjecture to higher dimensional varieties.

The Gross–Zagier formula provides an evidence to the B-SD conjecture for elliptic curves $E$ over $\mathbb{Q}$ when the $L$-function has a zero of order at most one. Let $f$ be the weight two newform associated to $E$ by the theorem of Wiles, Taylor–Wiles, and Breuil–Conrad–Diamond–Taylor. Let $\phi : X_0(N) \to E$ be a modular parameterization. Let $K$ be an imaginary quadratic extension of $\mathbb{Q}$, with discriminant $D$. Under suitable hypotheses, the theory of complex multiplication and the map $\phi$ allow us to define the Heegner point $y_K \in E(K)$. The Gross–Zagier formula is the following identity on the first order derivative of the base-changed $L$-function $L(E/K, s) = L(f/K, s)$ at the center $s = 1$ ([4, 6])

$$L'(f/K, 1) \frac{(f, f)}{(f, f)} = \frac{1}{|D|} \frac{(y_K, y_K)_{NT}}{\deg(\phi)},$$

where $(f, f)$ is the Petersson inner product. A similar formula, but for the central value of the $L$-function, was also discovered around the same time by Waldspurger [5].

What about higher order derivatives of $L$-function at the center? In [YZ15] a formula for arbitrary order derivative is proved for unramified cuspidal automorphic representation $\pi$ of $\text{PGL}_2$ over a function field $F = k(X)$, where $X$ is a curve over a finite field $k$. The $r$-th central derivative of the $L$-function (base changed along a quadratic extension $F'/F$) is expressed in terms of the self-intersection number of the Heegner–Drinfeld cycle $\text{Sht}^r_G$ (or rather its $\pi$-isotypic component) on the moduli stack $\text{Sht}^G_G$:

(2)

$$\mathcal{L}^{(r)}(\pi_{F'}, 1/2) \sim ([\text{Sht}^r_{\pi}], [\text{Sht}^r_{\pi}]).$$

The moduli stack $\text{Sht}^r_G$ is closely related to the moduli stack of Drinfeld Shtukas of rank two with $r$ modifications. One important feature of this stack is that it admits a natural fibration over the $r$-fold self-product $X^r$ of the curve $X$ over $\text{Spec} \ k$

$$\text{Sht}^G_G \longrightarrow X^r.$$ 

In the number field case, the analogous spaces only exist (at least for the time being) when $r \leq 1$. When $r = 0$, the moduli stack $\text{Sht}^0_G$ is the constant groupoid over $k$

(3)

$$\text{Bun}_G(k) \simeq G(F) \backslash (G(\mathbb{A}_F)/K),$$

where $\mathbb{A}_F$ is the ring of adèles of $F$, and $K$ a maximal compact open subgroup of $G(\mathbb{A})$. The double coset in the RHS of (3) remains meaningful for a number field $F$. When $r = 1$ the counterpart of $\text{Sht}^1_G$ in the case $F = \mathbb{Q}$ is the moduli stack of elliptic curves, which lives over $\text{Spec} \ (\mathbb{Z})$. Therefore the formula (2) can be viewed as a simultaneous generalization (for function fields) of the Waldspurger formula [5] (in the case of $r = 0$) and the Gross–Zagier formula [4] (in the case of $r = 1$). Moreover, there is a way to rewrite the RHS of the formula (2) so that it looks
just like (1). The formula (2) opens the possibility of relating higher derivatives of automorphic $L$-functions to geometric invariants in the function fields case.

The basic strategy of the proof of (2) is to compare two relative trace formulae (abbreviated as RTF), an “analytic” one for the $L$-functions, and a “geometric” one for the intersection numbers. The strategy of using RTF initiated by Jacquet in 1980s has been successful in related and similar questions on higher rank reductive groups when $r = 0$ (e.g., [7, 18]) and $r = 1$ (e.g., [16]).

The aim of the workshop is to carefully define the relevant objects that appear in the formula (2), especially the moduli stack of Shtukas and the Heegner–Drinfeld cycle; to review Jacquet’s RTF; and to sketch the geometric ideas used in the comparison of the two RTFs. The talks (except those providing general background) roughly correspond to various parts of the main reference [YZ15].

References


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