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Adaptive Algorithms

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Abstract. Overwhelming empirical evidence in computational science and engineering proved that self-adaptive mesh-generation is a must-do in real-life problem computational partial differential equations. The mathematical understanding of corresponding algorithms concerns the overlap of two traditional mathematical disciplines, numerical analysis and approximation theory, with computational sciences. The half workshop was devoted to the mathematics of optimal convergence rates and instance optimality of the Dörfler marking or the maximum strategy in various versions of space discretisations and time-evolution problems with all kind of applications in the efficient numerical treatment of partial differential equations.

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Introduction by the Organisers

The efficient numerical solution of PDEs requires an adaptive solution process in the sense that the discretization that is employed should depend in a proper, or rather optimal way on the solution. Since this solution is a priorily unknown, the adaptive algorithm has to extract the information where to refine from a sequence of increasingly more accurate computed numerical approximations, in a loop that is commonly denoted as **solve–estimate–mark–refine**. This loop has already been introduced by Babuska and co-workers in the seventies (eg. [1]), but is was not before 1997 that in [5], for linear elliptic PDEs in more than one dimensions, an adaptive finite element method (AFEM) was *proven* to converge to the exact solution.

Although this result meant an important step forward, it did not show that adaptive methods provide advantages over non-adaptive methods. These advantages are apparent from practical results, which often even show that adaptivity is paramount to be able to solve certain classes of complicated problems. This theoretical gap was closed some years later in [2, 6, 4], where it was shown that asymptotically AFEM converges with the *best possible rate* among all possible choices of the underlying partitions.

These initial results were restricted to model second order linear elliptic PDEs, discretized with standard conforming finite elements of a fixed polynomial degree, but they were a starting point for many researchers for proving similarly strong results for much larger classes of problems and discretizations. An abstract framework based on four assumptions ('axioms') (A1)-(A4), which identifies the basic mechanisms that underly the convergence and rate optimality proofs, and with that foster further applications was recently given in [3].

It was the purpose of this workshop to bring together experts that contributed to the recent developments in this field in order to exchange ideas, to stimulate further collaborations, and to identify or to make first steps in tackling remaining open challenges as *hp*-adaptivity, and optimal adaptive methods for instationary problems to name two important ones.

Moving to the contents of the in total 20 presentations, the aforementioned axioms do not not cover AFEMs with so-called separate marking for controlling data oscillation, as required for mixed and least squares problems. In her talk, Hella Rabus presented an even more general framework that applies to both collective and separate marking strategies.

The AFEM convergence theory was initially developed for symmetric positive definite (elliptic) problems, whereas later it was shown that possibly non-symmetric compact perturbations could be added. In his talk, Michael Feischl presented his recent ideas about how the framework could be extended to include strongly non-symmetric problems.

The application of the axioms for examples with non-conforming finite element schemes was discussed for the discrete reliability (A3) by Dietmar Gallistl and for boundary element schemes by Markus Melenk in terms of inverse estimates for the stability (A1) and (A2).

The Simons-professor Jun Hu presented three recent results on mixed finite element methods: an innovative class of novel mixed finite element methods with pointwise symmetric stress approximations in elasticity, an abstract framework for the convergence of adaptive algorithms, and preconditioners for the resulting system. He pointed out that there exist no convergence results for the Hellinger-Reissner formulation even in linear elasticity.

Joscha Gedicke presented the first robust a posteriori error estimators towards such an adaptive strategy and some numerical illustrations, where –to the surprise of the experienced audience– the bulk parameter in the Dörfler marking needed to be significantly smaller than 1/2. This and the size of the crucial bulk parameter in the framework of the aforementioned axioms showed the limitations of our

current understanding. Theoretical estimates in a worst case scenario suggest that it needs to be unrealistically small in comparison with the overall empirical knowledge of the workshop's participants. Speculations arose whether this implies that the current understanding of the optimal convergence rates leave out some undiscovered essentials. In the discussion, Carsten Carstensen also pointed out some difficulties in the treatment of vertices at the traction boundary in elasticity caused by the fact that the new and older symmetric stress approximations have nodal degrees of freedom whose naive approximation leads to contradictions even in simple benchmark examples and enforce a pre-asymptotic effect. The relaxed atmosphere in Oberwolfach allowed discussions amongst the experts and indeed sorted out this difficulty during the week, which will be visible in future research e.g. in ongoing projects of the SPP 1748 of the German research foundation.

Using techniques developed in the context of AFEM for eigenvalue problems, Alan Demlow demonstrated his proof of convergence and optimality of AFEM for harmonic forms. A major technical difficulty was the non-nestedness of the trial spaces under refinements. Another interesting class of non-standard problems is that of linear operator equations posed on Banach spaces. For those problems, Kris van der Zee presented a generalization of the Petrov-Galerkin methods with optimal test spaces, which methods in the generalized setting become nonlinear.

The class optimality result proven for AFEM says that if a solution can be approximated at a certain algebraic rate s, then the sequence of approximations produced by AFEM converges with that rate. Gantumur Tsogtgerel spoke about the question which functions can be approximated at rate s, characterized in terms of their smoothness. He presented new subtle results on the relation between approximation classes and classical smoothness spaces as Besov spaces.

Nearly all convergence results of AFEM are for bulk chasing, also known as the Dörfler marking strategy. An exception is given by a recent paper of Diening, Kreuzer and Stevenson in which, for the Poisson problem in two dimensions, even instance optimality is proven for an AFEM with a modified maximum marking strategy. In her talk, Mira Schedensack presented a generalization of this result to both Poisson and Stokes problems discretized with nonconforming Crouzeix Raviart finite elements.

Parameter dependent PDEs with a possibly infinite number of parameters are nowadays studied intensively. They arise for example by replacing a random coefficient field in a PDE by its Karhunen-Loeève expansion. In order to cope with the curse of dimensionality, one considers sparse (polynomial) expansions, or low rank approximations. In his talk, Wolfgang Dahmen presented an optimally converging adaptive solver in either of both formats. He showed that depending on the model either of the formats can realize the desired tolerance with the smallest number of terms.

For linear elliptic PDEs, it is known that AFEM does not only converge optimally in terms of the number of unknowns, but also in terms of the computational cost. For the latter it is needed that the exact solution of the arising Galerkin problems is replaced by an optimal iterative solution within a sufficiently small

relative tolerance. Dirk Praetorius spoke about a generalization of this result to nonlinear strongly monotone operators. As a first step he showed convergence of an AFEM with an iterative solver based on a Picard iteration. A related topic was discussed by Lars Diening. He presented a new algorithm of Kačanov type that can be used as an iterative solver of the nonlinear Galerkin problems that arise from the p-Laplacian.

So far most results about convergence of AFEM are for stationary (elliptic) problems. It can be foreseen that convergence and perhaps optimality of adaptive methods for time-dependent problems, in particular those of parabolic type, will an important topic in the coming years. In his talk, Omar Lakkis presented a posteriori error bounds for fully discrete Galerkin time-stepping methods using elliptic and time reconstruction operators.

The solution of a nonlinear parabolic problem may blow up in finite time. Emmanuil Georgoulis presented a conditional a posteriori bound for a fully-discrete first order in time implicit-explicit interior penalty discontinuous Galerkin in space discretization of a non self-adjoint semilinear parabolic PDE with quadratic nonlinearity. When used in a space-time adaptive algorithm to control the time step lengths and the spatial mesh modifications, the method detects and converges to the blow-up time without surpassing it.

Adaptive methods based on piecewise polynomial approximation of fixed degree can give at best algebraic convergence rates. Spectral- or hp finite element methods can yield even exponential convergence rates. Claudio Canuto showed how a convergent hp-afem can be turned into an instance optimal hp-afem by the addition of coarsening. An hp-adaptive tree algorithm that is perfectly suited for this task was presented by Peter Binev.

Having a p-robust a posteriori estimator is instrumental for the design of an hp-afem. In his talk, Martin Vohralík showed that the so-called equilibrated flux estimator, that is known to p-robust in two dimensions, is also p-robust in three dimensions. Serge Nicaise showed that this kind of error estimator is reliable and efficient for a magnetodynamic harmonic formulation of Maxwell's system.

Marco Verani considered adaptive spectral methods. He showed that exponential convergence rates can be achieved even without coarsening by the application of an 'aggressive' dynamic marking strategy.

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