Abstract. The Calculus of Variations is subject with a long and distinguished history, a great deal of diverse current activity, and close connections to other fields such as geometry and mathematical physics. The July 2016 workshop on the Calculus of Variations presented research that resolved long-standing conjectures, shed new light on classical results, pointed toward new research directions, and displayed progress on a range of aspects, classical and otherwise, of the Calculus of Variations.


Introduction by the Organisers

The workshop the Calculus of Variations featured 22 talks that presented research on a variety of topics connected to variational problems, including minimal or constant mean curvature surfaces, inequalities and symmetry, gradient and other flows, among others. This research was motivated by questions in geometry, analysis, statistical mechanics, data science, partial differential equations, and materials science, among other fields. The workshop was attended by 50 participants, of whom 18 were graduate students or postdoctoral fellows. It was organised by Simon Brendle (New York), Alessio Figalli (Zürich), Robert Jerrard (Toronto), and Neshan Wickramasekera (Cambridge).

Among the major highlights of the meeting, Jean Dolbeault spoke about striking work on the sharp constants in the critical and sub-critical Caffarelli-Kohn-Nirenberg inequalities. These results complete a long effort of many researchers to characterize exactly when extremals are radially symmetric. Another high point
of the conference was the talk of Guido de Philippis on the singular structure of Radon measures that belong to the kernel of a linear differential operator. This work sheds a new light on classical results in geometric measure theory, such as Alberti’s rank-1 theorem, and has various other important applications, including a weak converse to Rademacher’s Theorem. Another striking work presented at the conference was Aaron Naber’s result that establishes, in great generality, energy quantization and rectifiability of the defect measure associated to sequences of Yang-Mills connections.

Surfaces of constant or vanishing mean curvature are a subject at the heart of the calculus of variations, and were the focus of a number of talks. Brian White discussed which sets can occur as curvature blow-up sets of sequences of embedded minimal disks, thereby providing a partial converse to a deep theorem of Colding and Minicozzi. Costante Bellettini presented a regularity and compactness theory for stable constant-mean-curvature hypersurfaces that generalises earlier results for stable minimal hypersurfaces; this new theory is formulated in the full generality of codimension 1 integral varifolds and gives sharp regularity conclusions making no hypothesis on the singular set beyond two necessary structural conditions. Otis Chodosh reported on recent work on the uniqueness of large isoperimetric surfaces in asymptotically flat 3-manifolds. Andrea Mondino discussed the existence of optimal shapes for the isoperimetric-isodiametric inequality; this involves minimizing the product of surface area and radius subject to a volume constraint. Spencer Becker-Kahn presented results on the asymptotic behavior of two-valued Lipschitz minimal graphs of arbitrary dimension and codimension that are not assumed to satisfy any stability condition. Eleonora Cinti discussed quantitative flatness results and perimeter estimates for nonlocal minimal surfaces in low dimensions.

A number of talks addressed questions related to inequalities, sharp constants, symmetry, and stability. Among the highlights on these topics are the talk by Francesco Maggi which discussed sharp stability results for the euclidean concentration inequality and droplet formation in statistical mechanics, and the talk by Brian Krummel which discussed stability for Almgren’s isoperimetry principle, giving sharp estimates on the Fraenkel asymmetry and Hausdorff distance between the unit sphere in $(n+1)$-dimensional Euclidean space and a closed $n$-dimensional hypersurface with mean curvature at most $n$.

The Calculus of Variations is intimately connected to the study of gradient flows, and through them to other geometric evolution problems. Michael Struwe spoke about a variety of results concerning a supercritical nonlinear heat equation connected to long-standing problems in minmax theory. These included a novel monotonicity formula, small data global well-posedness in an optimal space, and some results about blow for large data. Yoshi Tonegawa presented recent work that, in the case of hypersurfaces, substantially strengthens Brakke’s foundational results on existence of weak solutions of the mean curvature flow for rough initial data. John Lott described the construction of a singular Ricci flow, obtained as a limit in a suitable sense of Ricci flow with surgeries on increasingly small scales,
partially answering a question of Perelman. Peter Topping presented the elegant proof of a new sharp $L^1 - L^\infty$ smoothing estimate for Ricci flow on surfaces, together with a parallel result for the logarithmic fast diffusion equation in 2 dimensions. Pei-Ken Hung presented new results about the asymptotic behavior of inverse mean curvature flow in hyperbolic space. Maria Colombo spoke about an extension of the DiPerna-Lions theory to the case of vector fields with fast growth, which she uses to establish existence of weak solutions of the Vlasov-Poisson equation for general initial data.

Striking new developments connected to various classical issues in the calculus of variations were presented in several talks. Filippo Santambrogio discussed a $\Gamma$-convergence result that establishes the validity of a new phase field approximation to the Steiner problem in the plane, and hence of associated numerical algorithms. Radu Ignat presented very refined results that derive a reduced free energy characterizing the interaction of domain walls in a critically-scaled nonlocal variational problem arising in micromagnetics. Susanna Terracini spoke about strong partial regularity results for shape optimization problems involving a combination of eigenvalues and a volume constraint.

Finally, connections between the calculus of variations and stochastic analysis appeared in a couple of very interesting talks. Charles Smart described the continuum limit of an algorithm that involves “convex hull peeling” for random point clouds. And in a very different direction, Robert Haslhofer presented deep results on Ricci curvature and martingales, showing that bounded Ricci curvature may be characterized in terms of a generalized Bochner formula on path space.

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Workshop: Calculus of Variations

Table of Contents

Michael Struwe (joint with Simon Blatt)
An optimal local well-posedness result for the supercritical Lane-Emden heat flow ........................................... 1949

Maria Colombo (joint with Luigi Ambrosio, Alessio Figalli)
Flow of nonsmooth vector fields and the Vlasov-Poisson system ... 1952

Filippo Santambrogio (joint with Matthieu Bonnivard, Antoine Lemenant)
Phase-field approximation of the Steiner problem. .................. 1955

Costante Bellettini (joint with Neshan Wickramasekera)
Stable constant mean curvature varifolds: regularity and compactness theory in codimension one .................. 1957

John Lott (joint with Bruce Kleiner)
Ricci flow through singularities ........................................ 1960

Eleonora Cinti (joint with Joaquim Serra, Enrico Valdinoci)
Quantitative flatness results and BV estimates for nonlocal minimal surfaces ................................................. 1961

Charles K. Smart (joint with Jeff Calder)
The limit shape of convex peeling ........................................ 1964

Francesco Maggi (joint with Eric A. Carlen, Alessio Figalli, Connor Mooney)
Sharp stability for the Euclidean concentration inequality and droplets formation in statistical mechanics .................... 1965

Pei-Ken Hung (joint with Mu-Tao Wang)
Asymptotic behavior of the inverse mean curvature flows in the hyperbolic spaces ................................................. 1968

Brian White
Helicoid-Like Minimal Surfaces ........................................... 1971

Radu Ignat (joint with Roger Moser)
Interaction energy of domain walls of logarithmically decaying tails in a nonlocal variational model ............................. 1974

Robert Haslhofer (joint with Aaron Naber)
Ricci curvature and martingales ........................................... 1977

Jean Dolbeault (joint with Maria J. Esteban, Michael Loss and Matteo Muratori)
Symmetry by flow .......................................................... 1980
Guido De Philippis (joint with Filip Rindler)
   On the structure of $\mathcal{S}$-free measures and applications. 1983

Andrea Mondino (joint with Emanuele Spadaro)
   On a isoperimetric-isodiametric inequality 1987

Otis Chodosh (joint with Michael Eichmair, Yuguang Shi, Haobin Yu)
   The isoperimetric problem for large volumes in asymptotically flat 3-manifolds 1990

Spencer T. Becker-Kahn (joint with Neshan Wickramasekera)
   Singularities of Minimal Two-Valued Graphs 1991

Brian Krummel (joint with Francesco Maggi)
   Isoperimetry with upper mean curvature bounds and sharp stability estimates 1994

Peter M. Topping (joint with Hao Yin)
   Sharp local smoothing estimates for the Ricci flow on surfaces 1997

Susanna Terracini (joint with Dario Mazzoleni and Bozhidar Velichkov)
   Regularity of the optimal sets for spectral functionals 1998

Yoshihiro Tonegawa (joint with Lami Kim)
   Existence theorem on the mean curvature flow starting from closed rectifiable set 2001