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Mini-Workshop: Singular Curves on $K3$ Surfaces and Hyperkähler Manifolds

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Abstract. The workshop focused on Severi varieties on $K3$ surfaces, hyperkähler manifolds and their automorphisms. The main aim was to bring researchers in deformation theory of curves and singularities together with researchers studying hyperkähler manifolds for mutual learning and interaction, and to discuss recent developments and open problems.

Mathematics Subject Classification (2010): Primary: 14H10, 14H20, 14H51, 14C20, 14J28, 14J50; Secondary: 14B07, 14E30.

Introduction by the Organisers

The workshop was attended by 15 participants with broad geographic and thematic representation. Its main aim was to bring together researchers in deformation theory of curves and singularities, especially working on Severi varieties of singular curves on $K3$ surfaces, together with researchers studying hyperkähler manifolds and their automorphisms.

Severi varieties take their name from the mathematician who introduced them at the beginning of last century. Let S be a smooth complex projective surface and $|D|$ a linear system on S containing smooth irreducible curves. The *Severi variety of δ-nodal curves* $V^S_{|D|,\delta} \subseteq |D|$ is defined as the locally closed subset of $|D|$ parametrizing irreducible curves with only δ nodes as singularities. Curves on smooth surfaces, their moduli and their enumerative geometry have been fundamental topics of algebraic geometry from the beginning of the previous century until today, thanks to the contribution of Severi, Segre, Zeuthen, Albanese, Enriques, Castelnuovo, Zariski, Arbarello, Cornalba, Harris, Shustin, Greuel and many others. An important breakthrough was made by Harris [18], who proved that Severi varieties of nodal plane curves are irreducible, as stated by Severi. Some years later, Kontsevich and Manin [23], by using Gromow-Witten theory, computed the degree of the Severi variety of rational plane curves. Their formulas were generalized by Caporaso and Harris [10], who found a recursive formula for the degree of Severi varieties of nodal plane curves of any genus, using only classical techniques. Later on, great progress was made in the study of the enumerative geometry of $V^S_{|D|,\delta}$, by among others Pandharipande, Vakil, Ran, Göttsche, Yau, Zaslow, Vainsencher, Tzeng and Thomas. Although a lot of work has been made on Severi varieties, many interesting problems remain open, especially in the case of $K3$ surfaces, as explained in the abstracts of **Ciliberto–Flamini** and **Dedieu**.

At the same time, the Brill-Noether theory of smooth curves on $K3$ surfaces has received a lot of attention in the last couple of decades, from the seminal papers of Lazarsfeld and Green [24, 17] to the more recent works on the Green conjecture and divisors on the moduli space of curves of Voisin, Farkas, Popa and Aprodu [26, 25, 14, 1]. Very recently, two conjectures about syzygies of curves, the *Green-Lazarsfeld secant conjecture* and the *Prym-Green conjecture* were (essentially) solved by Farkas and Kemeny in [12, 13] using curves on $K3$ surfaces, and an account of this is given in **Kemeny**'s abstract. Similarly, two outstanding conjectures by Wahl were established in [2], where it is proved that a Brill-Noether-Petri curve of genus ≥ 12 lies on a polarised $K3$ surface or on a limit of such if and only if the Wahl map for C is not surjective. An account of related open problems is made in **Sernesi**'s abstract.

The recent paper [11] starts the study of Brill-Noether theory of singular curves on a $K3$ surface S. Besides its intrinsic interest, the study is related to Mori theory of hyperkähler manifolds: indeed, curves on S with normalizations carrying pencils of degree k define rational curves on the Hilbert scheme $S^{[k]}$ of k points on the surface, one of the few examples known (together with its deformations) of hyperkähler manifolds. The other known examples are Albanese fibers of Hilbert schemes of points on abelian surfaces, called *generalized Kummer varieties*, (and their deformations), as well as two examples of O'Grady in dimensions 6 and 10. We recall that a (compact) *hyperkähler manifold* (or *irreducible holomorphic symplectic manifold*) is a simply-connected compact complex Kähler manifold X such that $H^0(X, \Omega^2_X)$ is spanned by a nowhere degenerate two-form. The interest in hyperkähler manifolds stems from Bogomolov's decomposition theorem for compact, complex Kähler manifolds with trivial canonical bundle in the 70s: up to finite étale cover they all decompose into products of Calabi-Yau, hyperkähler manifolds and tori. The birational geometry of hyperkähler manifolds is determined by their rational curves; in particular, rational curves determine their nef and ample cones, just like for $K3$s. Many years of research on this topic, passing in particular through several works and conjectures of Hassett and Tschinkel, culminated recently in the work of Bayer and Macrì [5] using *Bridgeland stability*, which determines (up to numerical computations) the extremal rays of the Mori cone of the Hilbert schemes of points on a $K3$ surface.

Despite recent advances by different methods, the study of curves on $K3$ or abelian surfaces with normalizations carrying special pencils still seems to be the most efficient way of concretely producing rational curves on hyperkähler manifolds. The results in [11] were recently extended to abelian surfaces in [21]. Some consequences of the results in [11, 21] on the birational geometry of the associated hyperkähler manifolds are obtained in [22] and the results and some open problems are given in **Knutsen**'s abstract.

Many of the recent results on singular curves on $K3$ (and abelian) surfaces have been proved by degenerating the surfaces. It is therefore natural to ask whether one can find similar degenerations of hyperkähler manifolds, as is done in **Galati**'s abstract, which also gives a brief account on the $K3$ case.

Another way of producing rational curves on $S^{[k]}$ is through automorphisms, as in e.g. [15]: the idea is to start with a special $K3$ surface such that $S^{[k]}$ contains a family of rational curves not present on the general projective deformation of it, use an automorphism of $S^{[k]}$ to produce another family of rational curves, and prove that the latter can be preserved under deformation. This is an interesting point of view, but one needs automorphisms of $S^{[k]}$ *not* coming from automorphisms of S, i.e. non-natural, and at the moment only one such example is known: the involution of Beauville on $S^{[2]}$ when S is a quartic. Thus one is in need of new such constructions. But the construction of new non-natural automorphisms on $S^{[k]}$ and more generally on other hyperkähler manifolds is an interesting and very active research topic on its own. The interest in automorphisms of hyperkähler manifolds has grown tremendously the last years. The foundational work on $K3$ surfaces by Nikulin, Mukai and Morrison was followed by classification results of Sarti with coauthors [3, 4, 16] and the recent work of Huybrechts [20]. Finally, the study of non-symplectic automorphisms on $K3$ surfaces has found a recent application in the study of Chow groups of $K3$ surfaces in particular in relation to the study of rational curves and the Bloch-Beilinson conjecture [19, 20]. Very little is known in higher dimensions, again there are results of Sarti, Boissière and coauthors [6, 7, 8, 9]. The abstract of **Boissière** gives an overview of results on automorphisms of special hyperkähler manifolds; more precise results and some open problems are formulated in the abstracts of **Camere** and **Cattaneo**, concerning existence of automorphisms and moduli spaces.

The abstracts of **Lehn**, **Saccà** and **Markushevich** explain other fundamental topics related to hyperkähler manifolds such as the construction of new manifolds, computation of Hodge numbers and Lagrangian fibrations. Finally, the abstract of **Ohashi** explains results on the automorphism group of Enriques surfaces and curve configurations. The study of the automorphism group of Enriques surfaces is very natural when studying automorphisms of $K3$ surfaces.

To promote interaction, the participants were asked to focus their talks on background results and open problems. Most talks were given in the first two days of the workshop to have time to discuss the proposed problems. We present the abstracts in chronological order and end with a few lines about the discussed open questions.

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Table of Contents