High-Resolution Mathematical and Numerical Analysis of Involution-Constrained PDEs

Organised by
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Abstract. Partial differential equations constrained by involutions provide the highest fidelity mathematical models for a large number of complex physical systems of fundamental interest in critical scientific and technological disciplines. The applications described by these models include electromagnetics, continuum dynamics of solid media, and general relativity. This workshop brought together pure and applied mathematicians to discuss current research that cuts across these various disciplines’ boundaries. The presented material illuminated fundamental issues as well as evolving theoretical and algorithmic approaches for PDEs with involutions. The scope of the material covered was broad, and the discussions conducted during the workshop were lively and far-reaching.


Introduction by the Organisers

The workshop High-Resolution Mathematical and Numerical Analysis of Involution-Constrained PDEs, organised by Bruno Després (Paris), Michael Dumbser (Trento), James Kamm (Los Alamos), and Manuel Torrilhon (Aachen), brought together 25 participants from institutes in France, Germany, Italy, the Netherlands, Norway, Russia, Switzerland, and the United States. The topics discussed by these researchers spanned an unusually broad spectrum of mathematics, including theoretical analysis, numerical analysis, and computational studies of involution-constrained PDEs. These topics can be characterized as a subset of the field of mimetic methods for PDEs, with specific focus on the algorithms that satisfy
the involutions constraining the allowable solutions. The workshop began with an overview talk by Jim Kamm, discussing the genesis of the workshop, the scope of its topics, and a brief discussion of numerical approaches to intrinsically satisfy the involution of Eulerian hyperelasticity.

Underpinning any computational approach to the solution of PDEs are theoretical examinations of numerical methods and the underlying structure of the equations. Snorre Christiansen presented the former, while the latter were discussed by Marc Gerritsma, Holger Heumann, and Allen Robinson, each of whom gave specific examples highlighting the relevance of exterior calculus, differential geometry, and algebraic topology in this field.

The Maxwell equations of electromagnetics provide the archetypal physics-based example of involution-constrained PDEs. Here, the involution is the constraint $\nabla \cdot \mathbf{B} = 0$ (where $\mathbf{B}$ is the magnetic induction), which is a necessary condition for physically admissible solutions. A closer examination of the nature of the equations of electromagnetics reveals a deep mathematical structure of the continuous PDEs. Careful algorithmic construction and numerical analysis is required to ensure that this structure is preserved in the corresponding numerical approximations. Topics in to this field were the basis of two presentations given by Alain Bossavit. One important application of electromagnetics is in the modeling of magnetohydrodynamic (MHD) phenomena, which was the focus of the presentations of Dinshaw Balsara, Bruno Després, and James Rossmanith. The MHD equations combine a particular limiting case of the Maxwell equations, combined with the conservation equations for material flow (fluid or solid). The proper formulation of the governing equations here remains an open issue, which has deep implications for both the continuous and discrete mathematics. Solutions of the MHD equations can display extremely complicated structures, the accurate computational approximation of which depends exquisitely on the numerical methods and, in particular, observation of the crucial involution constraint. Such accurate simulation is essential when considering flow instabilities, which not only provide a sensitive indicator of numerical solution quality but also represent a high-consequence physical phenomenon in experimental and industrial facilities, such as the proposed ITER reactor. For the latter application, closely related to MHD is the set of equations used to model the plasma physics relevant to near-wall regions in tokamaks. Aspects of the mathematical and numerical modeling associated with such edge plasma phenomena were addressed in the presentations of Emmanuel Franck, Boniface Nkonga, Ahmed Ratnani, and Eric Sonnendrücker.

The other primary component of the MHD model is given by the balance equations governing material motion. These conservation laws, generically referred to as the equations of hydrodynamics, govern not only fluid behavior but also condensed matter phenomena such as elasticity. Research of the last 30+ years has convincingly demonstrated the important role that high-resolution numerical methods play in the computational solution of these equations, even in the presence of flow discontinuities such as shock waves. The relation of high-resolution methods
to involution-constrained problems of compressible flow was discussed in the presentations of Christiane Helzel and Phil Roe. François Bouchut discussed the particular case of conservation laws applied to shallow water flow on a sphere, Roger Kaeppeli talked about the application of specific involution-conserving schemes to PDEs modeling astrophysical phenomena, and Michael Dumbser presented a new class of high order numerical schemes applicable to general systems of conservation laws. Likewise, the equations for hyperelastic material response are constrained by an involution, which is readily shown, in the Eulerian frame, to be the constraint $\text{curl} \, \mathbf{G} = 0$ (where $\mathbf{G}$ is the inverse deformation gradient). Related topics were discussed by Sergey Gavrilyuk, Gilles Kluth, Ilya Peshkov, and Evgeniy Romenskii.

Preservation of involution constraints is also required during the so-called data transfer process, in which the numerical solution to a set of PDEs is mapped from one computational mesh to another. This aspect is of broad-ranging interest in computational mathematics and was discussed in the presentations of Pavel Bochev and Misha Shashkov.

Summary of Topics Discussed

After the formal presentation program of the workshop, Bruno Després chaired a session in which the week’s topics were summarized. Here, we provide groups of presenters, grouped according to the primary perspective or initial motivation of their presented work.

**Differential geometry / exterior calculus / algebraic topology - inspired approaches:** A. Bossavit, S. Christiansen, M. Gerritsma, H. Heumann, A. Robinson, E. Sonnendrücker.

**Regular numerical methods for nonlinear elasticity:** B. Després, S. Gavrilyuk, G. Kluth, I. Peshkov, E. Romenskii

**High-order numerical methods:** M. Dumbser, C. Helzel, J. Rossmanith

**New nonlinear solvers:** D. Balsara, F. Bouchut, R. Kaeppeli, P. Roe

**Large nonlinear systems with preconditioners:** E. Franck, B. Nkonga, A. Ratnani

**Conservative data-transfer (remapping):** P. Bochev, M. Shashkov

Main Results and Important New Developments

There were several discussions during the workshop, each of which engendered energetic exchanges. The following themes were distilled from the discussions.

1. A key concept is that involutions help one understand the meaning of the equation vis-à-vis the physics.
2. Workshop participants were interested in constructive approaches to the solution of involution-constrained PDEs, i.e., in how to turn theoretical ideas into numerical methods.
3. An open question in several fields is, “How are we sure that we have the proper involutions?” Examples of this issue are seen in both MHD and nonlinear elasticity.
4. What is the relation between satisfying involutions and higher order methods? Should one or the other be foremost? Are lower order methods with involution constraints sufficient?
We give below a brief list of a few of the “hot topics” as drawn from the workshop
discussions.

**The role of exterior calculus:** in the numerical methods was subject to
spirited debate. Whereas the merit of this formulation for electromagnet-
ics (where it is both physically and mathematically illuminating) is not
questioned, the promise of this approach remains elusive for, e.g., hydro-
dynamics simulations. A key virtue of this approach is that it necessarily
involves the separation of topology from geometry in problems, and there
was speculation that this formalism might yet prove helpful for nonlinear
problems where the numerics necessary to preserve solution structure is
not yet known. This hopeful view was counterbalanced by the pragmatic
concerns that exterior calculus methods for other equations (1) must be
shown to exist and, if so, (2) should be evaluated as to their usefulness.

**The role of Riemann solvers:** in compressible flow and MHD was a topic
of vigorous dialogue. The role of Riemann solvers as a sub-grid model
providing “just the right amount” of dissipation was discussed, and the
place for approximate multidimensional Riemann solvers was debated.

**Appropriate discretization schemes:** was a theme that spanned both of
the above topics. The concept of the topology of the discretization was a
recurring theme, tempered by the notion that one should not overburden
a computational mesh into satisfying *ipso facto* certain constraints.

Bruno Després, Michael Dumbser, James Kamm, Manuel Torrilhon
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