Mathematisches Forschungsinstitut Oberwolfach

Report No. 39/2013

DOI: 10.4171/OWR/2013/39

# Multiscale and High-Dimensional Problems

Organised by
Albert Cohen, Paris
Wolfgang Dahmen, Aachen
Ronald A. DeVore, College Station
Angela Kunoth, Paderborn

28 July – 3 August 2013

Abstract. High-dimensional problems appear naturally in various scientific areas, such as PDEs describing complex processes in computational chemistry and physics, or stochastic or parameter-dependent PDEs leading to deterministic problems with a large number of variables. Other highly visible examples are regression and classification with high-dimensional data as input and/or output in the context of learning theory. High dimensional problems cannot be solved by traditional numerical techniques, because of the so-called *curse of dimensionality*.

Such problems therefore amplify the need for novel theoretical and computational approaches, in order to make them, first of all, tractable and, second, offering finer and finer resolutions of relevant features. Paradoxically, increasing computational power serves to even heighten this demand. The wealth of available data itself becomes a major obstruction. Extracting essential information from complex structures and developing rigorous models to quantify the quality of information in a high dimensional context leads to tasks that are not tractable by existing methods.

The last decade has seen the emergence of several new computational methodologies to address the above obstacles. Their common features are the nonlinearity of the solution methods as well as the ability of separating solution characteristics living on different length scales. Perhaps the most prominent examples lie in adaptive grid solvers, tensor product, sparse grid and hyperbolic wavelet approximations and model reduction approaches. These have drastically advanced the frontiers of computability for certain problem classes in numerical analysis.

This workshop deepened the understanding of the underlying mathematical concepts that drive this new evolution of computation and promoted the exchange of ideas emerging in various disciplines about the handling of multiscale and high-dimensional problems.

*Mathematics Subject Classification (2010):* 16xx (Numerical Analysis and Scientific Computing).

## Introduction by the Organisers

Inherently high-dimensional problems appear naturally in various scienitific areas, such as the Fokker-Planck and the Schrödinger equations as examples of PDEs describing complex processes in computational chemistry and physics, or stochastic or parameter-dependent PDEs leading to deterministic problems with a large number of variables. Complex scientific models like climate models, of turbulence, fluid structure interaction, nanosciences and reliability control, demand finer and finer resolution in order to increase reliability. This demand is not simply solved by increasing computational power. Indeed, higher computability even contributes to the problem by generating wealthy data sets for which efficient organization principles are not available. Extracting essential information from complex structures and developing rigorous models for quantifying the quality of information is an increasingly important issue. This manifests itself through recent developments in various areas.

The mathematical methods emerging to address these problems have to exploit in a much more subtle way the structure of the problem in order to extract the necessary information. They have several common features including the question whether the underlying objects have a sufficiently small information content, how this content might be accessible through certain sparse representations, the nonlinearity of the solution methods as well as the ability of separating solution characteristics living on different length scales. Having to deal with the appearance and interaction of local features at different levels of resolution has, for instance, brought about spatially adaptive methods as a key methodology that has advanced the frontiers of computability for certain problem classes in numerical analysis.

A related but different concept for managing the interaction of different length scales centers on wavelet bases and multilevel decompositions. In the very spirit of harmonic analysis they allow one to decompose complex objects into simple building blocks that again support analyzing multiscale features. However, for high-dimensional problems, wavelets are not the only answer although they may serve as a reference for classical representations.

While this ability was exploited first primarily for treating *explicitly* given objects, like digital signals and images or data sets, the use of such concepts for recovering also *implicitly* given objects, like solutions of partial differential or boundary integral equations, has become a major recent focus of attention. The close marriage of discretization, analysis and the solution process based on *adaptive* wavelet methods has led to significant theoretical advances as well as new algorithmic paradigms for linear and nonlinear stationary variational problems. Through thresholding and best $N$-term approximation based on wavelet expansions, concepts from nonlinear approximation theory and harmonic analysis become practically manageable. In our opinion, these ideas have opened promising perspectives not only for signal and image processing but also for the numerical analysis of

differential and integral equations covering, in particular, such operator equations with high dimensional deterministic or stochastic parameter dependence. For the latter, smooth dependence of the solution on the parameters can be exploited for achieving highly efficient approximations.

These various concepts have developed relatively independently of one another. Our previous Oberwolfach Workshops "Wavelet and Multiscale Methods" held in July 2004, August 2007 and August 2010 sought to bring various disciplines utilizing multiscale techniques together by inviting leading experts and young emerging scientists in areas that rarely interact. Those workshops not only accelerated the advancement of nonlinear and multiscale methodologies but also provided beneficial cross–fertilizations to an array of diverse disciplines which participated in the workshop, see the Oberwolfach Reports 34/2004, 36/2007 and 33/2010. Among the several recognizable outcomes of the workshops were: (i) the emergence of compressed sensing as an exciting alternative to the traditional sensing-compression paradigm, (ii) fast online computational algorithms based on adaptive partition for mathematical learning, (iii) clarification of the role of coarsening in adaptive numerical methods for PDEs, (iv) injection of the notion of sparsity into stochastic models to identify computational paradigms that are more efficient than Monte Carlo techniques.

One of the main objectives of this workshop was to foster synergies by the interaction of scientists from different disciplines resulting in more rapid developments of new methodologies in these various domains. It also served to bridge theoretical foundations with applications, such as mathematical finance, quantum chemistry, signal and image processing, complex fluid flows. Examples of conceptual issues that were advanced by our workshop were:

- adaptive and nonlinear multilevel methods for high-dimensional PDEs;
- multilevel and high dimensional meshless methods;
- Convergence of low-rank tensor approximations to solutions of high-dimensional PDEs;
- adaptive treatment of nonlinear and time–dependent variational problems;
- interaction of different scales under nonlinear mappings;
- convergence theory and analysis for model reduction;
- extension of model reduction methods to unsymmetric, indefinite and singularly perturbed problems;
- polynomial interpolation and adaptive quadrature in high dimensions;
- uncertainty quantification and stochastic inversion;
- regularity of solutions to stochastic differential equations;
- high dimensional dynamical systems for modeling flocking and consensus formation;
- harmonic analysis and frames on general compact domains and manifolds.

The workshop had large success in bringing together researchers from diverse disciplines who rarely see one another but have common interest in high dimensional problems and their numerical treatment. This led to a lot of interesting discussions and new ideas that will surely be pursued over the next years.

In summary, we feel that the conceptual similarities that occur in these diverse application areas suggest a wealth of synergies and cross–fertilization. These concepts are in our opinion not only relevant for the development of efficient solution methods for large scale and inherently high-dimensional problems but also for the formulation of rigorous mathematical models for quantifying the extraction of essential information from complex objects in many dimensions.

As in the previous workshops, the participants are experts in areas like nonlinear approximation theory (e.g., Cohen, Dahmen, DeVore), statistical learning theory (e.g., Kerkyacharian, Picard), tensor approximations (Grasedyck, Hackbusch, Oseledets, Schneider, Yserentant), sparse grids (e.g., Harbrecht, Schwab), finite elements (e.g., Oswald, Stevenson), convergence of adaptive methods (e.g. Dahlke, Stevenson), spectral methods (e.g., Canuto), harmonic analysis and wavelets (e.g., Cohen, Dahmen, Petrushev, Schneider), strongly nonlinear DPEs in high dimensions (e.g., Süli), numerical fluid mechanics and conservation laws (e.g., Müller, Popov, Tadmor), inverse problems (e.g., de Mol), multiscale modeling (e.g., Müller, Ohlberger, Tadmor), parameter-dependent PDE-constrained control problems (e.g., Kunoth), model reduction and reduced basis functions (e.g., Dahmen, Grepl, Ohlberger, Urban, Wojtaszczyk), stochastic PDEs and regularity of their solutions (e.g., Dahlke, Larsson, Nobile, Schwab), modeling flocking and consensus formation processes (e.g. Fornasier, Tadmor) and tractability of multivariate problems (Wozniakowski).

## Workshop: Multiscale and High-Dimensional Problems

## Table of Contents

Roman Andreev
*Space-time sparse tensor Petrov-Galerkin discretization of parabolic evolution equations* . . . . . 2187

Markus Bachmayr (joint with Wolfgang Dahmen)
*Adaptive near–optimal rank tensor approximation for high–dimensional operator equations* . . . . . 2190

Peter Binev
*Instance optimality for hp-type approximation* . . . . . 2192

Francesca Bonizzoni (joint with Fabio Nobile, Daniel Kressner and Christine Tobler)
*Low-rank techniques applied to moment equations for the stochastic Darcy problem with lognormal permeability* . . . . . 2194

Dietrich Braess (joint with Holger Dette)
*A multivariate approximation problem in statistics* . . . . . 2197

Moulay Abdellah Chkifa (joint with Albert Cohen and Christoph Schwab)
*High-dimensional adaptive sparse polynomial interpolation and application for parametric and stochastic elliptic PDE's* . . . . . 2199

Stephan Dahlke (joint with P.A. Cioica, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter and R.L. Schilling)
*Adaptive wavelet methods for SPDEs: Theoretical analysis and practical realization* . . . . . 2200

Nira Dyn (joint with Michael S. Floater)
*Multivariate polynomial interpolation on monotone sets* . . . . . 2203

Martin A. Grepl (joint with Mark Kärcher)
*Reduced basis methods for parametrized optimal control problems* . . . . . 2204

Wolfgang Hackbusch
*Numerical tensor calculus and tensorisation* . . . . . 2207

Markus Hansen
*N-term approximation and Besov regularity for parametric elliptic PDEs* 2209

Helmut Harbrecht (joint with Michael Peters and Markus Siebenmorgen)
*Multilevel quadrature for elliptic stochastic partial differential equations* 2212

Gerard Kerkyacharian (joint with P. Petrushev, Y. Xu)
*Analysis on convex subset of a Riemannian manifold and classical polynomial approximation* . . . . . 2215

Annika Lang (joint with Christoph Schwab)
*Isotropic Gaussian random fields on the sphere* . . . . . 2216

Jean-Marie Mirebeau
*Adaptive, hierarchical cones of convex functions* . . . . . 2219

Christian Mollet
*Stability of Petrov-Galerkin discretizations: Application to the weak space-time formulation for parabolic PDEs* . . . . . 2222

Siegfried Müller (joint with Wolfgang Dahmen, Giulia Deolmi)
*Effective boundary conditions for compressible flows over rough boundaries* . . . . . 2225

Mario Ohlberger (joint with F. Albrecht, M. Drohmann, B. Haasdonk, P. Henning, S. Kaulmann, and B. Schweizer)
*Model reduction for multiscale problems* . . . . . 2228

Pencho Petrushev (joint with Shai Dekel, Gerard Kerkyacharian, and George Kyriazis)
*Compactly supported frames for spaces of distributions in the framework of Dirichlet spaces* . . . . . 2231

Bojan Popov (joint with Jean-Luc Guermond and Orhan Mehmetoglu)
*Maximum principle and entropy consistency for numerical approximations of nonlinear hyperbolic conservation laws* . . . . . 2233

Reinhold Schneider (joint with M. Bachmayr)
*Convergence of dynamical low rank approximation in hierarchical tensor formats* . . . . . 2234

Christoph Schwab (joint with Claudia Schillings)
*Sparse quadrature approach to Bayesian inverse problems* . . . . . 2237

Rob Stevenson (joint with Christian Kreuzer and Lars Diening)
*Instance optimality of the adaptive maximum strategy* . . . . . 2241

Endre Süli (joint with Iain Smears)
*Analysis and approximation of Hamilton–Jacobi–Bellman equations with Cordès coefficients* . . . . . 2243

Eitan Tadmor
*Clustering and consensus in collective dynamics* . . . . . 2245

Gerrit Welper (joint with Wolfgang Dahmen and Christian Plesken)
*Reduced basis methods for transport dominated problems* . . . . . 2248

Przemek Wojtaszczyk
*Properties of greedy algorithm used in reduced basis method* . . . . . 2249

Henryk Woźniakowski
*Tractability of multivariate problems* . . . . . 2251

Harry Yserentant (joint with Ludwig Gauckler)
*Regularity and approximability of the solutions to the chemical master equation* . . . . . 2251