Progress in Surface Theory

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Abstract. Over the last 30 years global surface theory has become pivotal in the understanding of low dimensional global phenomena. At the same time surface geometry became a platform on which seemingly different areas of mathematics – such as geometric and topological analysis, integrable systems, algebraic geometry of curves, and mathematical physics – coalesce to produce far reaching ideas, conjectures, methods and results. The workshop hosted talks on the resolutions of famous conjectures in surface geometry, including the Willmore conjecture, and on exciting new progress in the understanding of moduli spaces of special surface classes.

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Introduction by the Organisers

The workshop Progress in Surface Theory brought together 26 participants including PhD and PostDoc researchers. The 21 talks were evenly spread over the duration of the workshop and generally lasted 60 minutes, except for shorter 50 or 30 minute presentations by junior researchers. The schedule allowed ample time for discussions and ongoing and emerging collaborations between participants. The workshop also hosted a well attended and lively problem session. Incidentally, the workshop Geometric Knot Theory took place during the same week and the morning session on Wednesday was held jointly with talks by Andre Neves and Joel Hass.

A central theme of the meeting was the study of moduli spaces of special surface classes, including Willmore, minimal, and constant mean/Gauss curvature surfaces, by different methods such as geometric analysis, integrable systems and
algebraic curve theory. Researchers in these fields benefit significantly from mutual interactions and the workshop provided a stimulating atmosphere for such exchanges.

A number of lectures addressed the recent resolutions of famous conjectures in global surface geometry. Andre Neves gave a series of extraordinary talks on applications of min-max theory, including the solution of the Willmore conjecture and a solution of a conjecture about the minimal Möbius energy link configuration. A further highlight was the resolution of a generalized version of the Lawson conjecture (first mentioned by Pinkall and Sterling), stating that the only embedded constant mean curvature tori in the 3-sphere are the rotational ones. The proof, which uses non-collapsing arguments developed for the mean curvature flow and ideas from Brendle’s proof of the Lawson conjecture, was beautifully explained in a lecture by Ben Andrews. Using techniques from algebro-geometric integrable systems, Martin Schmidt characterized the moduli space of constant mean curvature cylinders of finite spectral genus in the 3-sphere in terms of their hyper-elliptic spectral curves. When applied to the special situation of constant mean curvature tori this description provides a conceptually different proof of the (generalized) Lawson conjecture. Finally, Wilhelm Klingenberg outlined elements for a proof of the Caratheodory conjecture (in joint work with Brendan Guilfoyle) by combining Lagrangian geometry and mean curvature flow.

A significant advance in the understanding of higher genus constant mean curvature surfaces in the 3-sphere was presented by Sebastian Heller, who outlined a program to understand their moduli via Abelianization of flat connections. A first glimpse of the progress made was a description of Lawson’s genus 2 minimal surface (and its constant mean curvature deformations) in terms of an explicit family of Fuchsian connections over the Riemann sphere and the (as yet numerical) solution of its accessory parameter problem.

There were several additional lectures at the interface of geometric analysis and integrable systems. In one such lecture, Francis Burstall explained how to associate a Lagrangian density to a map into the space of lines (known already to Darboux) and characterized the “harmonic maps” in this setting. Special cases include the mean curvature sphere congruence of Willmore surfaces and a recently studied functional on Lagrangian surfaces (arising from a cubic differential) in the complex projective plane. Christoph Bohle used the Weierstrass representation of a conformal immersion given by solutions of the Dirac operator (associated to the induced spin bundle of the immersion) with mean curvature potential to construct constant mean curvature disks with prescribed (bi-normal) boundary values. On a related topic Ulrich Pinkall defined a gradient flow of the Willmore functional on the submanifold of mean curvature potentials giving rise to conformal immersions. Thus, rather than flowing the geometric object (the conformal immersion) directly the flow acts on an infinitesimal invariant of the surface. By design the flow preserves the conformal structure of the immersion and its fixed points are constrained Willmore surfaces. It is known that constrained Willmore tori can be obtained from spectral curves of finite genus. Lynn Heller characterized all
constrained Willmore tori of spectral genus $g \leq 2$ in terms of constrained elastic curves on the 2-sphere. She also conjectured that the minimizers for the Willmore energy in conformal classes near the Clifford torus should be among those constrained Willmore Hopf tori. Drawing upon an analogue between the Riemann mapping theorem and the Plateau problem for minimal surfaces, Laura Desideri outlined an approach to solve the Plateau problem with analytic boundary via the universal Schlesinger system. The latter is a generalization of the classical Schlesinger system arising in the study of isomonodromic deformations of Fuchsian connections. Atsufumi Honda studied (extrinsically) flat fronts, that is flat surfaces with special singularities, in Euclidean space and the 3-sphere and also described a transformation theory of these surfaces. Finally, Mark Haskins reported on some recent progress in the construction of compact 7-manifolds with $G_2$-holonomy via twisted connected sums of asymptotical cylindrical Calabi-Yau 3-folds.

The 3-dimensional spaces of constant curvature are the classical target spaces for surface geometry. Recent years have seen continued interest in the study of surfaces in other 3-dimensional target spaces, especially those which are homogeneous. Besides intrinsic motivations for their study, one can apply rescaling arguments to construct special surfaces (e.g. minimal) in the classical target spaces. Josef Dorfmeister explained a loop group description for minimal surfaces in the 3-dimensional Heisenberg group. Joaquin Perez studied constant mean curvature surfaces in 3-dimensional Lie groups endowed with a left-invariant metric and related the isoperimetric problem to the Cheeger constant and the critical mean curvature of the ambient space. Considering periodic minimal surfaces as maps into tori, Toshihiro Shoda computed the index and nullity of families arising from the Abel maps of hyper-elliptic Riemann surfaces. Using the Lie quadric as a target space, Udo Hertrich-Jeromin discretized linear Weingarten surfaces (whose Gauss and mean curvature satisfy an affine relation) in any of the space forms (also with signatures) and indicated their integrable structure and transformation theory.

Three lectures addressed recent progress in the theory of isoparametric hypersurfaces in spheres. First, Hui Ma studied the Gauss maps of isoparametric hypersurfaces as special examples of minimal Lagrangian submanifolds in the complex quadric. For instance, the Gauss maps of homogeneous isoparametric hypersurfaces turn out to be Hamiltonian stable. In a related talk, Reiko Miyaoka interpreted the Karcher-Münzer-Ferus polynomials, which arise in the study of non-homogeneous isoparametric hypersurfaces, as moment maps for the Spin action. Finally, Anna Siffert used the relationship (given by the Gauss map) between hypersurfaces in spheres and Lagrangian submanifolds in the complex quadric to outline a structural approach to the classification problem of isoparametric hypersurfaces.

The talks gave a balanced view of current results and ongoing research in the field of differential geometry of surfaces. At the same time the lectures demonstrated that the tandem of geometric analytical and integrable systems techniques
can significantly deepen our understanding of the properties of special surface classes. In this sense, the workshop provided an ideal backdrop for the exchange of ideas, understanding of techniques, stimulation of collaboration, and development of new approaches in the field. For instance, it would not be surprising if the interior ball curvature estimates were more widely applicable and could lead to more general rigidity results, including a different proof of the Willmore conjecture; or if the integrable systems techniques combined with gradient flow methods could be used to give a more detailed picture (and perhaps even a proof) of a yet to be formulated constrained Willmore conjecture; or if the moduli of holomorphic and flat bundles over complex curves could provide the correct setting to obtain a complete picture of constant mean curvature surfaces of higher genus and with ends.
Workshop: Progress in Surface Theory

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